THE INCIRCLE HYPERBOLOID AND ELLIPSOID OF THE CONJUGATE PAVILLET TETRAHEDRA

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The Pavillet Tetrahedron is a unique orthocentric tetrahedron attached to a triangle called base triangle (vertices A, B, C). Its has numerous properties which can be used to prove classical triangle geometry theorems or, conversely, triangle geometry theorems can be used to prove some of its properties. It is built from the incircle ($\mathscr{C}_I$) of the base triangle (radius r). The incenter (I) is the apex of the tetrahedron, the other three vertices (A', B', C') form a triangle called the upper triangle.

In the paper we consider all triangles circumscribed to a fixed incircle. Hence the apex I is fixed and the vertices of the upper triangle A', B', C', which positions are function of the tangential distance of the vertices of the base triangle to the incircle ($AA' = AL = AM = x, \ldots$), describe a surface in $\mathbb{R}^3$. By symmetry about the plane of the incircle, two Pavillet tetrahedra $IA'B'C'$ and $I\bar{A}'\bar{B}'\bar{C}'$ can be built from a single base triangle. We denote H'' and $\bar{H}''$ their orthocenter and call them conjugate tetrahedra because we show that the links between these two tetrahedra are more than a simple symmetry.

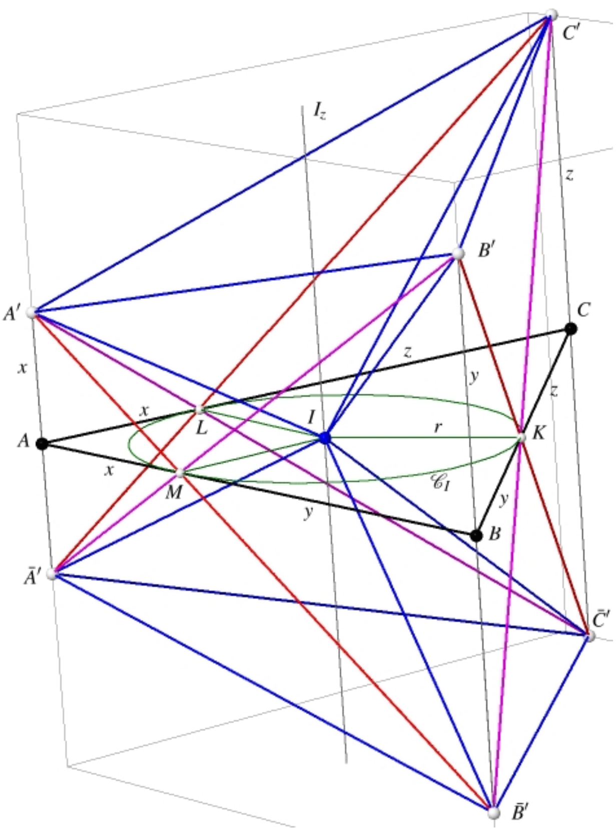

Figure 1: *The two reguli: $A'\bar{B}'$ and $\bar{A}'B'$.*

1 The hyperboloid

The simplest proof for the orthocentricity of the tetrahedron is the fact that the sum of the square of the length of a pair of opposite edges is a constant. For this proof we use Pythagora's Theorem twice and we easily get $IA'^2 = AA'^2 + AM^2 + IM^2 = 2x^2 + r^2$ because the triangle AMA' is right and *isosceles*. It implies that the slope of $A'M$ in the vertical plane $A'AB$ is $\frac{\pi}{4}$. We use the same property to get $C'B'^2 = 2y^2 + 2z^2$ which yields the result. It was noted in a former article that the same proof for the symmetric tetrahedron implied that the triplets $A'M\bar{B}'$ and similar are collinear. We conclude that we get the straight line supporting $B'K\bar{C}'$ from the one supporting $A'M\bar{B}'$ by a rotation about a vertical axis I_z going through the incenter and similarly for $C'L\bar{A}'$. This proves that the six vertices lie on a one sheet hyperboloid of revolution, with a vertical axis, equilateral, which has I for center of symmetry. Obviously, the second set of generating lines are the lines such as $A'L\bar{C}'$.

2 Tangent planes and polar properties

The plane of the upper triangle of one of the tetrahedra intersect the hyperboloid along a conic section hence the tangent planes to this section envelope a second order cone. We prove that the vertex of this cone is $\bar{H}''$, the orthocenter of the conjugate tetrahedron. This proof uses the fact that the segments $A'K, B'L$ and $C'M$ where KLM is the contact triangle of ABC are three altitudes of the tetrahedron. Then the paper considers the pencil of planes along a generating line. We use the fact that the cross ratio of the tangent planes is equal to the cross ratio of the points of tangency to add six more interesting points (linked to the Nobbs points) to this hyperboloid.

In a first paper, we proved that the incircle sphere, the sphere having the incircle as great circle, was a polar sphere of the orthocentric group I, A', B', C', H''. It implied that the orthocentric tetrahedron $H''A'B'C'$ was self-polar about this sphere. In this paper, we prove that the tetrahedra $H''A'B'C'$ and $\bar{H}''\bar{A}'\bar{B}'\bar{C}'$ are polar reciprocal about the hyperboloid.

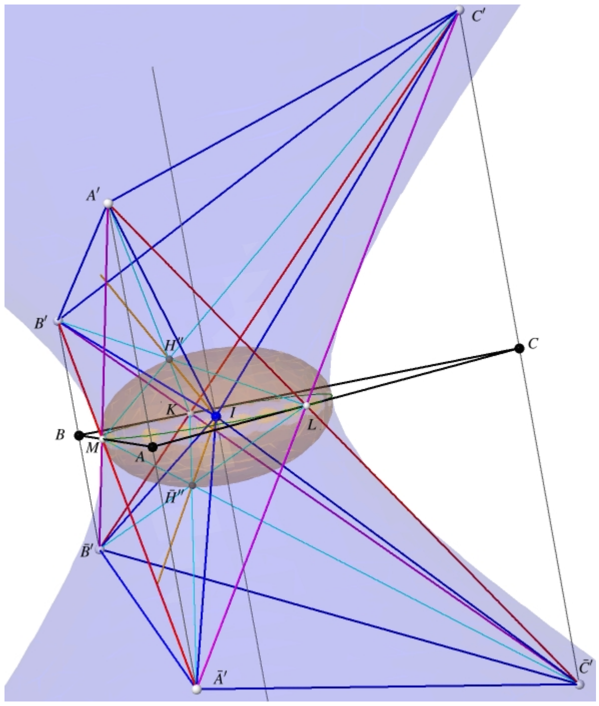

Figure 2: *The orthocenters lying on the ellipsoid inscribed in the throat circle of the hyperboloid.*

3 The incircle ellipsoid

Then, because the orthocenters play such an important role, we also examine the locus of the orthocenters when A, B, C describe the plane of the base triangle. This locus is easy to find because the orthogonal projection of the orthocenters on the plane of the base triangle is the Gergonne point of this triangle (G_e) and there is an invariant relation of the tetrahedron proved in the initial paper: $g^2 + 3n^2 = r^2$ where g is the distance IG_e and n the altitude of the orthocenter about the plane of the base triangle $n = H''G_e$. In cylindrical coordinates, it is the equation of an oblate ellipsoid of revolution ($x^2 + y^2 + 3z^2 = r^2$) which is inscribed in the throat circle of the hyperboloid ($x^2 + y^2 - z^2 = r^2$). Therefore, because, to any set of triangle vertices A, B, C correspond a single Gergonne point G_e and because, when A, B, C describe the outer part of the incircle, G_e describes the inner part, we see that, given a fixed circle, all the triangles circumscribed to this incircle have their conjugate Pavillet tetrahedra attached to these two tangent and complementary surfaces.

Keywords: Tetrahedron Hyperboloid Ellipsoid Revolution Gergonne Orthocenter

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ABSTRACT: The Pavillet Tetrahedron is a unique orthocentric tetrahedron attached to a triangle called base triangle. Its has numerous properties which can be used to prove triangle geometry theorems or, conversely, triangle geometry theorems can be used to prove its properties. It is built from the incircle of the base triangle; the incenter is called the apex of the tetrahedron, the other three vertices form a triangle called the upper triangle. In this paper we consider the incircle as given. Hence the apex of the tetrahedron is fixed and the vertices of the upper triangle, which positions are a function of the tangential distance of the vertices of the base triangle to the incircle, describe a surface in $\mathbb{R}^3$. While the vertices of the base triangle describe the plane of the incircle outside the incircle, the corresponding Gergonne point of this triangle describe the inner part of the incircle. Similarly, the orthocenter of the tetrahedron which project on the plane of the incircle as the Gergonne point of the base triangle describe a surface. We show that the vertices of the upper triangle lie on a one sheet equilateral hyperboloid of revolution while the orthocenter lies on an oblate ellipsoid of revolution inscribed in the throat circle of this hyperboloid. Moreover by symmetry two Pavillet tetrahedra can be built from a single base triangle. We call them conjugate tetrahedra because we show that the links between these two tetrahedra created by the hyperboloid is stronger than a simple symmetry.

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1 Introduction.

We recall that the orthocentric tetrahedron of a scalene triangle [3], named the Pavillet tetrahedron by Richard Guy and Gunther Weiss [6], is formed by drawing from the vertices A, B and C of a triangle ABC on the horizontal plane, called the base triangle three vertical segments $AA' = AM = x$, $BB' = BK = y$, $CC' = CL = z$ where KLM is the contact triangle of ABC. We then consider the tetrahedron $IA'B'C'$ (fig. 1). The three points A', B', C' form a triangle called the upper triangle and define a plane, the upper plane. The figure is rich in properties and various of them have been described in [3],[2] and [4].

We can build on both sides of the plane of the base triangle so the Pavillet tetrahedron of ABC, $IA'B'C'$ has a symmetric tetrahedron; we designate the second set of vertices and points by $\bar{A}', \bar{B}', \bar{C}'$ and call this pair of tetrahedra the conjugate tetrahedra of the triangle. The orthocenters of the tetrahedra will respectively be called H'' and $\bar{H}''$.

2 The Hyperboloid of revolution.

The nine points A', B', C', $\bar{A}'\bar{B}'\bar{C}'$ and K, L, M define a unique quadric surface.

Theorem 2.1 *The quadric surface defined by the six vertices of the upper triangles of the conjugate tetrahedra and the contact triangle of the base triangle is a one sheet equilateral hyperboloid of revolution centered at I with a vertical axis.*

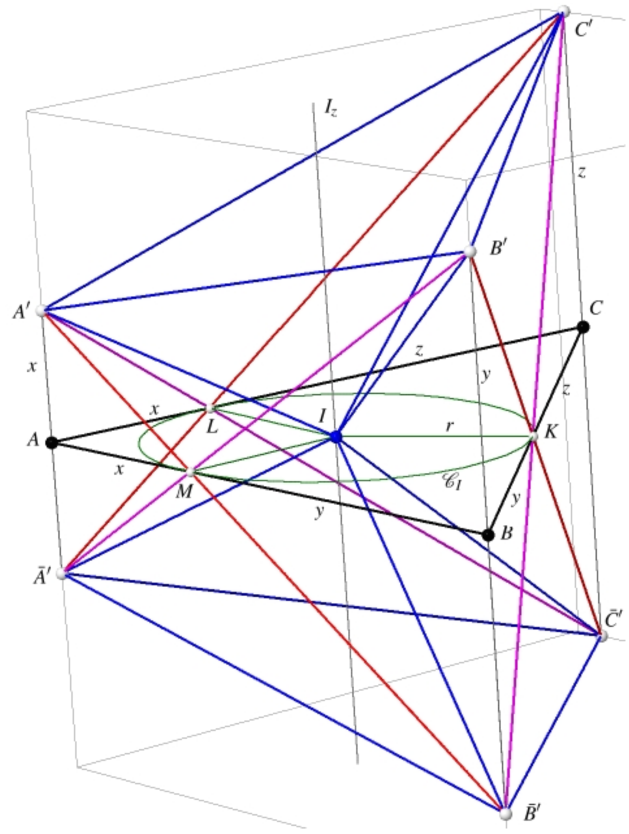

Figure 1: *The two reguli: $A'\bar{B}'$ and $\bar{A}'B'$.*

Proof.

The simplest proof for the orthocentricity of the tetrahedron is the fact that the sum of the square of the length of a pair of opposite edges is a constant. For this proof (in [3], fig. 1), we used Pythagora's Theorem twice and we easily got $IA'^2 = AA'^2 + AM^2 + IM^2 = 2x^2 + r^2$ because the triangle AMA' is right and *isosceles*. It implies that the slope of $A'M$ in the vertical plane $A'AB$ is $\frac{\pi}{4}$. We use the same property to get $C'B'^2 = 2y^2 + 2z^2$ which yields the result . It was noted in [3, §3 - corollary 3.2] that the same proof for the symmetric tetrahedron also implied that the triplets $A'M\bar{B}'$ and similar are collinear. We conclude that we get the straight line supporting $B'K\bar{C}'$ from the one supporting $A'M\bar{B}'$ by a rotation about a vertical axis I_z going through the incenter and similarly for the straight line supporting $C'L\bar{A}'$. This proves that the nine vertices lie on a one sheet hyperboloid of revolution, equilateral because the slope is $\frac{\pi}{4}$, which has I for center of symmetry and $\mathscr{C}_I$ for throat circle.

Clearly the three segments $A'L$, $B'M$ and $C'K$ lie on one of the set of straight lines generating the hyperboloid while the other regulus is the set to which belong $A'M$, $B'K$ and $C'L$.

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The way we defined the hyperboloid is linked only to the incircle $\mathscr{C}_I$ and does not depend in any way of the choice of the other components of the triangle. Note that if $\mathscr{C}_I$ is fixed, the choice of A, B, C is not totally arbitrary, we can choose the first vertex anywhere outside $\mathscr{C}_I$ and then a second one has to be chosen on one of the half-lines (not going through A) supported by the tangents to the incircle going through A. The third one is then constrained. But any point of the plane outside the incircle can be the vertex of a triangle tangent to $\mathscr{C}_I$. Therefore, we have proved that when a triangle has $\mathscr{C}_I$ for incircle, its Pavillet tetrahedron has the vertices of its upper triangle on a equilateral hyperboloid having $\mathscr{C}_I$ for throat circle.

In [3] we called the sphere having the incircle as great circle the *incircle sphere*, its equation relative to the incenter is:

$$\mathscr{S}_I : x^2 + y^2 + z^2 = r^2. \quad (1)$$

We are going to see that our set of two symmetric tetrahedra has properties very similar relative to the hyperboloid as one of them relative to the incircle sphere. Hence we call this hyperboloid the *incircle hyperboloid*; its equation with the same origin is:

$$\mathscr{H}_I : x^2 + y^2 - z^2 = r^2. \quad (2)$$

3 The tangent planes.

Theorem 3.1 *The tangent plane at any vertex of one of the upper triangle goes through the orthocenter of the conjugate tetrahedron.*

Proof.

We recall that the segments $A'K, B'L$ and $C'M$ are three altitudes of the orthocentric tetrahedron [3, §1]. Their intersection, H'', is such that its projection on the plane of the base triangle is G_e, the Gergonne point of the base triangle. This is valid for the conjugate tetrahedron. Now the

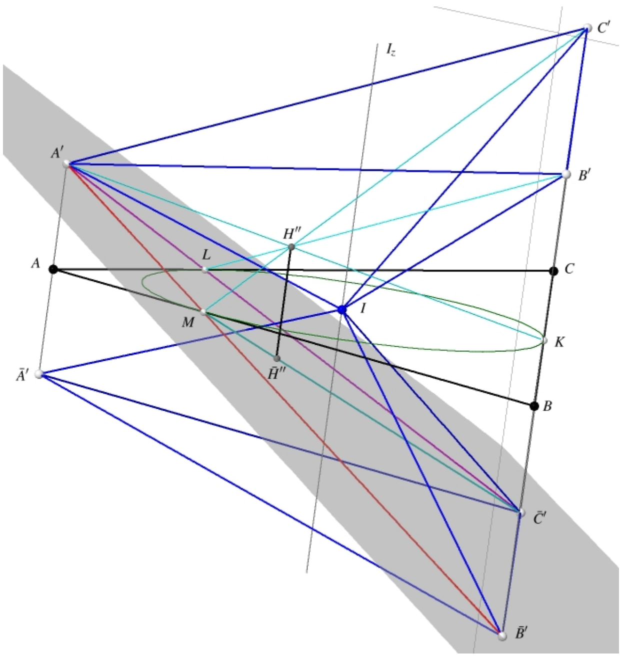


Figure 2: *The tangent plane at A'.*

tangent plane to the hyperboloid at one of these vertices is the plane define by both generatrices going through this vertex, e.g. the tangent plane at A' (fig. 2) is defined by the two lines $A'M$ and $A'L$ or, also, by $A'\bar{B}'$ and $A'\bar{C}'$ and therefore the line $\bar{C}'M$ lies on the tangent plane at A' but $\bar{C}'M$ being an altitude of the conjugate tetrahedron, $\bar{H}''$ lies on the tangent plane to the hyperboloid at A'.

■

The intersection of the hyperboloid by an upper plane is a conic and consequently the three tangent planes at the upper vertices of one of the tetrahedron belong to the envelope of the tangent cone to the hyperboloid through the orthocenter. Therefore,

Theorem 3.2 *The pole of the upper plane of one tetrahedron about the hyperboloid is the orthocenter of its conjugate.*

Remark 3.1 *Note that when the intersection of one of the upper plane with the hyperboloid is a parabola, the angle between the upper plane and the horizontal plane is $\frac{\pi}{4}$ which from [5] and [3, §7] is known to be the special case when the outer Soddy circle of ABC degenerates to a line.*

It is easy to check that the line of intersection of any two of the six tangent planes is either a generatrix of the hyperboloid, an altitude of one of the tetrahedron (excluding the one from the apex) or a line of support of the contact triangle.

3.1 Six more points.

We recall that both upper planes intersect the base plane along the Gergonne line of the base triangle [3, §1.3]. As in [1], we call the points of intersections of the sides of the base triangle with the sides of the contact triangle K_g, L_g, M_g, the Nobbs points; they are collinear and lie on the Gergonne line of the triangle. We notice that the tangent plane at one vertex intersects the horizontal plane along a line which supports the contact triangle, e.g. the tangent plane at A' intersects the base plane along LM and therefore intersects the Gergonne line at one of the Nobbs points, K_g such that $(B,C,K,K_g) = -1$.

Remark 3.2 *The base triangle has to be scalene for the general case; if it is isosceles, e.g. at C, then M is the midpoint of AB and therefore M_g, its harmonic conjugate is at infinity. If the* *triangle is equilateral, then the whole Gergonne line is at infinity.*

From the three Nobbs points we draw a vertical line. This line will intersect the lines of the generating sets; e.g., K_g lying on BC, the vertical line from K_g will intersect the generatrix $B'\bar{C}'$ at K_p and the generatrix $C'\bar{B}'$ at K_m. We get six more points on our hyperboloid ($K_p, K_m, L_p, L_m, M_p, M_m$), the upper and lower Nobbs points, and we look for the corresponding tangent planes.

We already know one line of this tangent plane, the corresponding generatrix. To get the second one, we call $\Pi_{B'}$ the tangent plane at B', $\Pi_{\bar{C}'}$ the tangent plane at $\bar{C}'$, Π_{K_p} the tangent plane at K_p, Π_K the tangent plane at K and this one is just the vertical plane $B'KC'$ because K lies on the throat circle of the hyperboloid.

To get Π_{K_p}, we start from $(B,C,K,K_g) = -1$ which implies that $(B',\bar{C}',K,K_p) = -1$. The hyperboloid being a non developpable ruled surface, we have

$$(B',\bar{C}',K,K_p) = -1 \Rightarrow \left(\Pi_{B'},\Pi_{\bar{C}'},\Pi_K,\Pi_{K_p}\right) = -1.$$

The pencil of these four planes is harmonic so its intersection by any line has to be harmonic. We examine the intersection of this pencil with the orthocenter line, the vertical line through G_e joining H'' and $\bar{H}''$. We already know that $\Pi_{B'} \cap H''\bar{H}'' = \bar{H}''$ and $\Pi_{\bar{C}'} \cap H''\bar{H}'' = H''$. Now $\Pi_K \cap H''\bar{H}'' = \infty_{Iz}$ because, at K, the tangent plane is vertical. Therefore the point of intersection of Π_{K_p} with $H''\bar{H}''$ has to be the harmonic conjugate of ∞_{Iz} about H'' and $\bar{H}''$, the midpoint of $H''\bar{H}''$. This point is the Gergonne point of the base triangle.

We have found that Π_{K_p}, the tangent plane at K_p, is formed by the lines $B'\bar{C}'$ and $G_eK = AK$ and we can get to the other ones by cyclical permutations.

We have also proved

Theorem 3.3 *The tangent planes to the hyperboloid at the upper and lower Nobbs points, intersect at the Gergonne point of the triangle which means that the Gergonne point of the base triangle is the pole about the hyperboloid of the vertical plane drawn through its Gergonne line.*

Remark 3.3 *The same result can be obtained considering the trace of the pencil* $\left(\Pi_{B'},\Pi_{\bar{C}'},\Pi_K,\Pi_{K_p}\right)$ *on the horizontal plane.*

4 Poles and polar properties.

In [3, §5], we proved that the incircle sphere, the sphere having the incircle as great circle, was a polar sphere of the orthocentric group formed by the vertices and orthocenter of the tetrahedron $IA'B'C'$. It implies that the orthocenter is the pole of the plane of the upper triangle with respect to this sphere. Here we have similarly proved Theorem 3.2.

It is shown in [3, §5, Theorem 5.4] that the line of orthocenters, $G_eH'' = \bar{H}''H''$, is the conjugate of the Gergonne line about the incircle sphere but this is also true for the hyperboloid because the conjugate of $H''\bar{H}''$ is the line of intersection of two polar planes of points lying on this line, e.g. H'' and $\bar{H}''$, i.e. the two upper planes which, by symmetry intersect along the Gergonne line.

Similarly, if we consider the two altitudes going through the apex I and therefore through the center of both quadrics (IH'', $I\bar{H}''$), the conjugate line has to be a line at infinity. It is the line at infinity of the corresponding upper plane if we consider the sphere, the line at infinity of the conjugate upper plane if we consider the hyperboloid.

Going back to the five points I, A', B', C', H'', it is an orthocentric group therefore the orthocentric tetrahedron $H''A'B'C'$ is self polar with respect to the incircle sphere. We compare with the polar tetrahedron of $H''A'B'C'$ with respect to the incircle hyperboloid. The polar plane of one of the

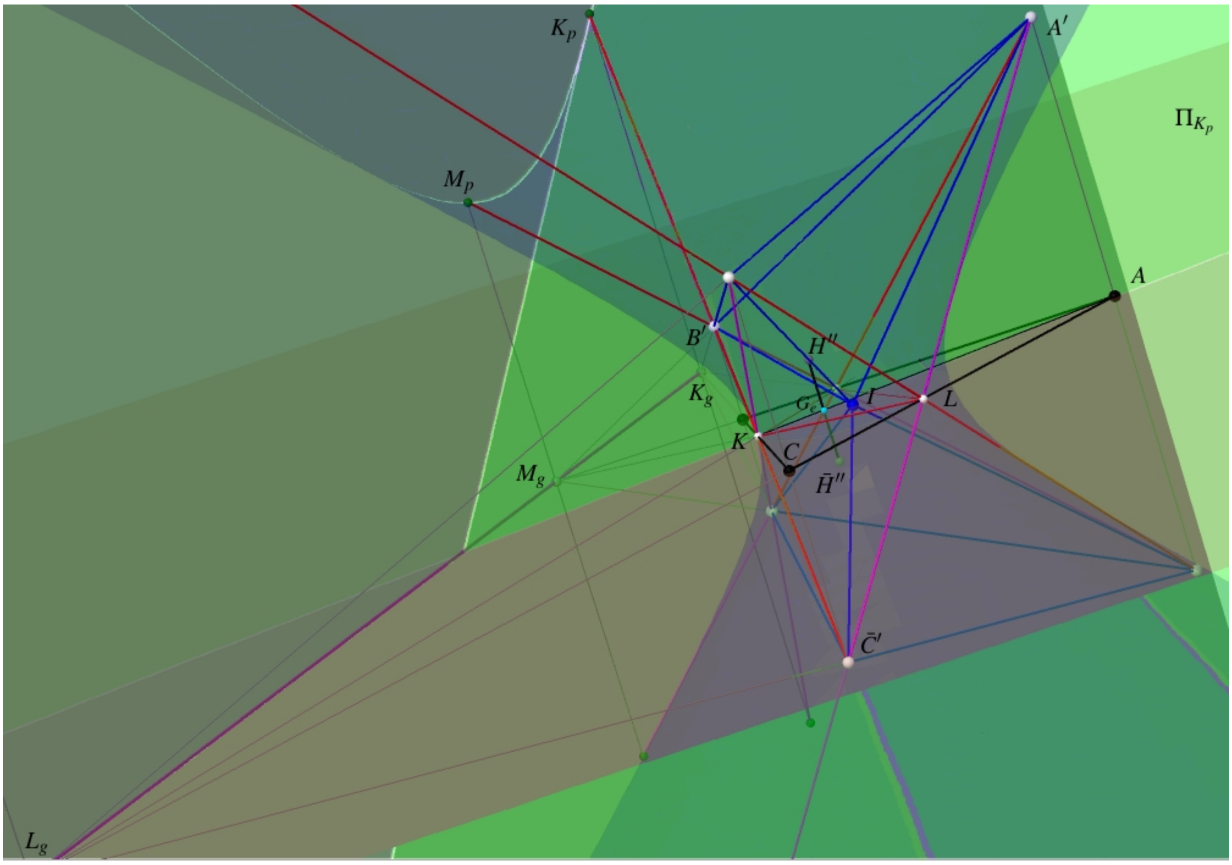


Figure 3: *The plane Π_{K_p} tangent to the hyperboloid at K_p intersects the horizontal plane along AK*

orthocenters, e.g. H'', is the upper plane of the conjugate tetrahedron defined by $\bar{A}', \bar{B}', \bar{C}'$ while the polar plane of a vertex of the upper triangle, A', B', C', is the tangent plane to the hyperboloid at this point so that the polar tetrahedron is formed by one secant and three tangent planes.

Now the tangent plane at A' is also defined by the three points $\bar{H}'', \bar{B}', \bar{C}'$ (Theorem 3.1, fig. 2) so that the polar tetrahedron of $H''A'B'C'$ with respect to the hyperboloid is its conjugate $\bar{H}''\bar{A}'\bar{B}'\bar{C}'$.

Theorem 4.1 *The tetrahedra $H''A'B'C'$ and $\bar{H}''\bar{A}'\bar{B}'\bar{C}'$ are self polar relative to the incircle sphere but polar reciprocal with respect to the incircle hyperboloid.*

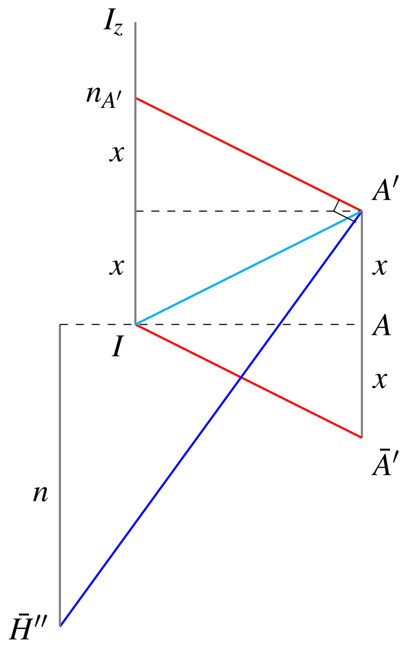
Therefore, the polarity about the incircle sphere keeps apart the elements of both tetrahedra while the polarity about the hyperboloid swaps them. It justifies the name of conjugate tetrahedra.

4.1 Normals to the hyperboloid at the fifteen points

We can also use the orthocentric group to find the normal to the hyperboloid at the vertices. As mentioned, the tetrahedron $\bar{H}''\bar{A}'\bar{B}'\bar{C}'$ is orthocentric and I is its orthocenter. Therefore $I\bar{A}'$ is orthogonal to the plane $\bar{H}''\bar{B}'\bar{C}'$ but this plane is the tangent plane to the hyperboloid at A' (fig. 2).

Therefore we have proved that the normal to the hyperboloid at one of the vertices of the tetrahedra is the parallel to the edge of the conjugate tetrahedra joining the apex to the corresponding conjugate vertex.

These normal will intersect the axis of the hyperboloid, the vertical line through the incenter I_z (fig. 2), because it is a surface of revolution. As shown on the following projection on the vertical plane IAA', the altitude of the point of intersection is twice the tangential distance of the vertex of the base triangle to the incircle ($x = AL = AM$) and $\bar{A}'A' = In_{A'} = 2x$.



To find the normal to the upper and lower Nobbs point, e.g. K_p, we can use the fact that the relationship between the normal to the hyperboloid on a generator (here $B', \bar{C}', K, K_p$) and the corresponding point is algebraic and one to one. Then the cross ratio is preserved and because the set $\left(B', \bar{C}', K, K_p\right)$ is harmonic then the intersection of the normals at these points with the axis of the hyperboloid is also harmonic. The normals at B' and $\bar{C}'$ are known from above and intersect the axis at $n_{B'}$ and $n_{\bar{C}'}$. The tangent plane at K being vertical, the normal is the line KI which intersects the axis I_z at I, therefore the normal at K_p intersects the axis at n_{K_p} harmonic conjugate of I about $n_{B'}$ and $n_{\bar{C}'}$. Hence, we can either construct n_{K_p} geometrically or compute its altitude z_{nk_p} with the equation

$$\left(z_{nk_p}, 0, 2y, -2z\right) = -1 \Rightarrow z_{nk_p} = \frac{4yz}{z-y}.$$

All other values can be deduced by simultaneous permutations and we have now all the values for the fifteen points found on the incircle hyperboloid. As seen above (Remark 3.2), if the triangle is isosceles, e.g. at C, $x = y$ then the corresponding point is at infinity.

5 The incircle ellipsoid.

We have seen that the orthocenters of the conjugate tetrahedra play an important role in this geometric figure and so it becomes interesting to examine the locus of the orthocenters, also a surface, when $A', B', C', \bar{A}', \ldots$ describe the hyperboloid.

This locus is easy to find because the orthogonal projection of the orthocenters on the plane of the base triangle is the Gergonne point of this triangle (G_e) and there is an invariant relation of the tetrahedron proved in [3, §6]: $g^2 + 3n^2 = r^2$ where r is the in-radius, g is the distance IG_e and n the altitude of the orthocenter about the plane of the base triangle $n = G_eH''$. Interpreting this relation in cylindrical coordinates about the incenter shows that this is the equation of an oblate ellipsoid of revolution

$$\mathscr{E}_I : x^2 + y^2 + 3z^2 = r^2, \tag{3}$$

which, again, is inscribed in $\mathscr{C}_I$, the throat circle of the incircle hyperboloid $\mathscr{H}_I$ (2).

Therefore, because, to any set of points A, B, C forming a triangle circumscribed about $\mathscr{C}_I$ correspond a single Gergonne point G_e and because, when A, B, C describe the outer part of the incircle, G_e describes the inner part, we see that, given a fixed circle, all the triangles which have

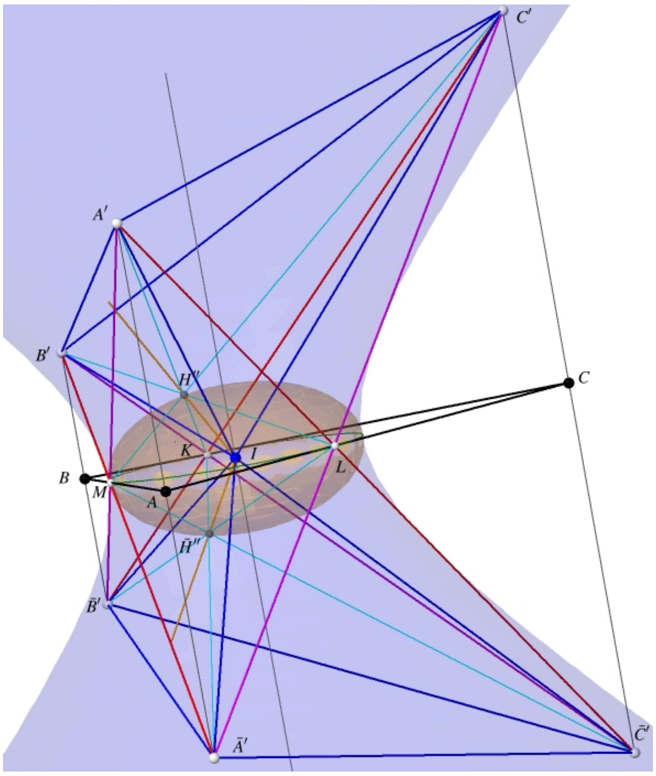


Figure 4: *The orthocenters lying on the ellipsoid inscribed in the throat circle of the hyperboloid.*

this circle as incircle have their conjugate Pavillet tetrahedra attached to these two tangent and complementary surfaces (fig. 4).

5.1 Tangent plane to the ellipsoid at the orthocenters.

Because we know the surface on which the orthocenters lie, we may look for the tangent planes to the ellipsoid at these two points. As usual we have more than one way to find it, let's use a pencil of quadric.

- The incircle sphere and incircle hyperboloid form a pencil of quadrics. Their base curve is the incircle, counted twice, and at each point of the incircle they are tangent.
- Hence the incircle ellipsoid belongs to this pencil. Of course, we could as well check that, using (1), (2) and (3), we have

$$2\mathscr{S}_I - \mathscr{H}_I = \mathscr{E}_I.$$

- The polar planes of a point with respect to a pencil of quadrics form a pencil of planes, we apply this to one of the orthocenters.
- The polar plane of an orthocenter about the incircle sphere is the corresponding upper plane [3, Theorem 5.2].
- The polar plane of this orthocenter about the incircle hyperboloid is the conjugate upper plane (cf. Theorem 3.2).
- Both planes intersect at the Gergonne line of the base triangle (§3.1, fig. 5).
- So the axis of this pencil of polar planes is the Gergonne line,
- and the polar plane of the orthocenter about the ellipsoid goes through the Gergonne line.
- Now the orthocenter lies on the ellipsoid so its polar plane about this quadric is the tangent plane at this point.

We have proved

Theorem 5.1 *The tangent planes to the incircle ellipsoid at the orthocenters intersect the horizontal plane along the Gergonne line of the base triangle.*

We note that the eccentricity of the meridian ellipse $x^2+3z^2=r^2$ of the incircle oblate ellipsoid is $e=\sqrt{\frac{2}{3}}$ and is a constant for any triangle or incircle.

6 Conclusions.

We have shown that for a given circle, all triangles circumscribed about this circle have the vertices

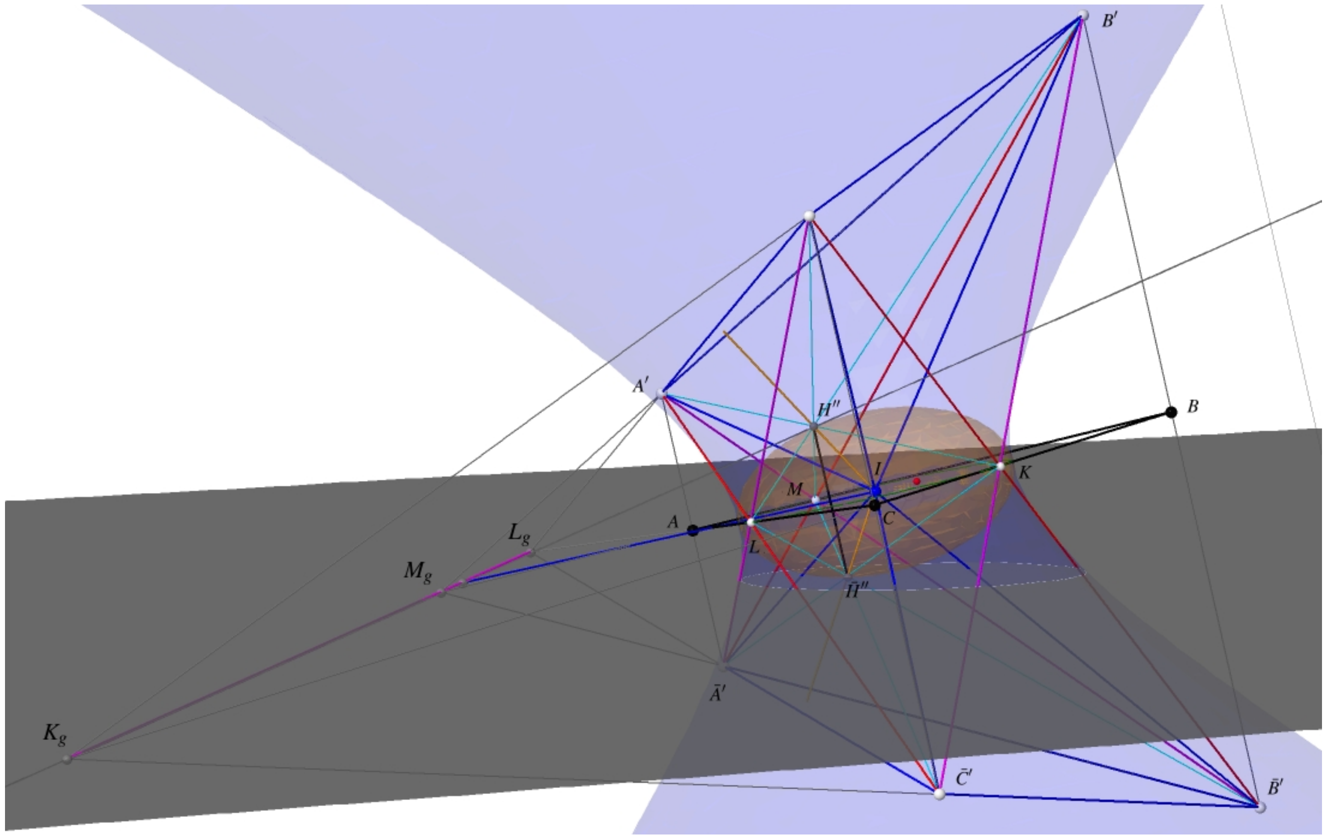


Figure 5: *The tangent planes to the ellipsoid at the orthocenters.*

of their conjugate Pavillet tetrahedra lying on a one sheet hyperboloid of revolution, equilateral, which admit the incircle as its throat circle. Their common apex is the center of symmetry of the hyperboloid. We also have shown that the tangent planes of five sets of three points of the hyperboloid (vertices of the upper triangles, vertices of the contact triangle, lower and upper Nobbs points) intersect on a common vertical line going through the Gergonne point of the base triangle. The polar properties with respect to the incircle sphere, the sphere inscribed in the throat circle of the hyperboloid, and with respect to the hyperboloid are either the same (orthocenters line) or reversed (4th altitudes, upper planes, orthocentric group). Finally the locus of the orthocenters of the conjugate set of tetrahedra is also given, an ellipsoid oblate part of the pencil of quadrics formed by the incircle sphere and hyperboloid. The Gergonne line of the base triangle lies on the tangent planes to the ellipsoid at the orthocenters.

We have found a great deal of properties to this new geometric figure and all the proofs are extremely simple.

Giving one more constraint to the triangle would replace this two parameters problem by a one parameter problem; the vertices and the orthocenters would describe a curve which would lie on one of these two surfaces. As a simple example, we can consider the degenerate case of the Soddy outer circle (Remark 3.1). In that case the orthocenter will be such that the distance $IG_e = \frac{r}{2}$, and therefore the orthocenters will describe two symmetric horizontal circles on the ellipsoid. This example and others deserve further studies. It was mentioned in [3] that Richard Guy had extended the properties of the Pavillet tetrahedron to the excircles so that another possible development are the *excircle hyperboloids*, the extension should be straightforward but the relationships between the four hyperboloids can reveal interesting properties.

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