THE INCIRCLE HYPERBOLOID AND ELLIPSOID OF THE CONJUGATE PAVILLET TETRAHEDRA

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The Pavillet Tetrahedron is a unique orthocentric tetrahedron attached to a triangle called base triangle (vertices A, B, C). Its has numerous properties which can be used to prove classical triangle geometry theorems or, conversely, triangle geometry theorems can be used to prove some of its properties. It is built from the incircle ($\mathscr{C}_I$) of the base triangle (radius r). The incenter (I) is the apex of the tetrahedron, the other three vertices (A', B', C') form a triangle called the upper triangle.

In the paper we consider all triangles circumscribed to a fixed incircle. Hence the apex I is fixed and the vertices of the upper triangle A', B', C', which positions are function of the tangential distance of the vertices of the base triangle to the incircle ($AA' = AL = AM = x, \ldots$), describe a surface in $\mathbb{R}^3$. By symmetry about the plane of the incircle, two Pavillet tetrahedra $IA'B'C'$ and $I\bar{A}'\bar{B}'\bar{C}'$ can be built from a single base triangle. We denote H'' and $\bar{H}''$ their orthocenter and call them conjugate tetrahedra because we show that the links between these two tetrahedra are more than a simple symmetry.

1 The hyperboloid

The simplest proof for the orthocentricity of the tetrahedron is the fact that the sum of the square of the length of a pair of opposite edges is a constant. For this proof we use Pythagora's Theorem twice and we easily get $IA'^2 = AA'^2 + AM^2 + IM^2 = 2x^2 + r^2$ because the triangle AMA' is right and *isosceles*. It implies that the slope of $A'M$ in the vertical plane $A'AB$ is $\frac{\pi}{4}$. We use the same property to get $C'B'^2 = 2y^2 + 2z^2$ which yields the result. It was noted in a former article that the same proof for the symmetric tetrahedron implied that the triplets $A'M\bar{B}'$ and similar are collinear. We conclude that we get the straight line supporting $B'K\bar{C}'$ from the one supporting $A'M\bar{B}'$ by a rotation about a vertical axis I_z going through the incenter and similarly for $C'L\bar{A}'$. This proves that the six vertices lie on a one

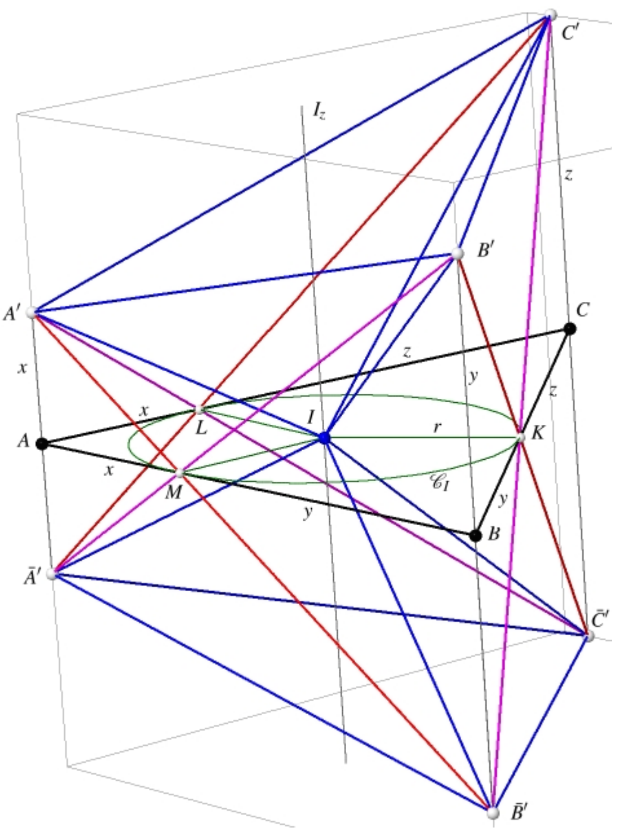

Figure 1: *The two reguli: $A'\bar{B}'$ and $\bar{A}'B'$.*

sheet hyperboloid of revolution, with a vertical axis, equilateral, which has I for center of symmetry. Obviously, the second set of generating lines are the lines such as $A'L\bar{C}'$.

2 Tangent planes and polar properties

The plane of the upper triangle of one of the tetrahedra intersect the hyperboloid along a conic section hence the tangent planes to this section envelope a second order cone. We prove that the vertex of this cone is $\bar{H}''$, the orthocenter of the conjugate tetrahedron. This proof uses the fact that the segments $A'K, B'L$ and $C'M$ where KLM is the contact triangle of ABC are three altitudes of the tetrahedron. Then the paper considers the pencil of planes along a generating line. We use the fact that the cross ratio of the tangent planes is equal to the cross ratio of the points of tangency to add six more interesting points (linked to the Nobbs points) to this hyperboloid.

In a first paper, we proved that the incircle sphere, the sphere having the incircle as great circle, was a polar sphere of the orthocentric group I, A', B', C', H''. It implied that the orthocentric tetrahedron $H''A'B'C'$ was self-polar about this sphere. In this paper, we prove that the tetrahedra $H''A'B'C'$ and $\bar{H}''\bar{A}'\bar{B}'\bar{C}'$ are polar reciprocal about the hyperboloid.

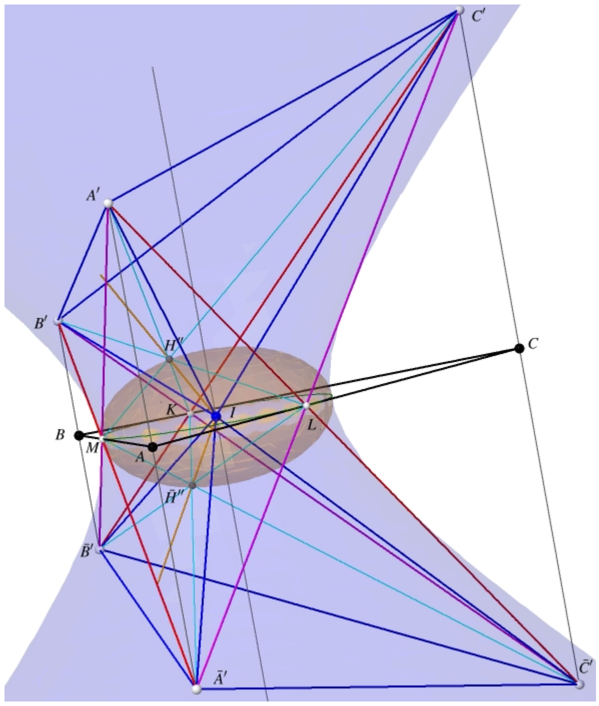

Figure 2: *The orthocenters lying on the ellipsoid inscribed in the throat circle of the hyperboloid.*

3 The incircle ellipsoid

Then, because the orthocenters play such an important role, we also examine the locus of the orthocenters when A, B, C describe the plane of the base triangle. This locus is easy to find because the orthogonal projection of the orthocenters on the plane of the base triangle is the Gergonne point of this triangle (G_e) and there is an invariant relation of the tetrahedron proved in the initial paper: $g^2 + 3n^2 = r^2$ where g is the distance IG_e and n the altitude of the orthocenter about the plane of the base triangle $n = H''G_e$. In cylindrical coordinates, it is the equation of an oblate ellipsoid of revolution ($x^2 + y^2 + 3z^2 = r^2$) which is inscribed in the throat circle of the hyperboloid ($x^2 + y^2 - z^2 = r^2$). Therefore, because, to any set of triangle vertices A, B, C correspond a single Gergonne point G_e and because, when A, B, C describe the outer part of the incircle, G_e describes the inner part, we see that, given a fixed circle, all the triangles circumscribed to this incircle have their conjugate Pavillet tetrahedra attached to these two tangent and complementary surfaces.

Keywords: Tetrahedron Hyperboloid Ellipsoid Revolution Gergonne Orthocenter