COAXAL PENCIL OF CIRCLES AND SPHERES IN THE PAVILLET TETRAHEDRON

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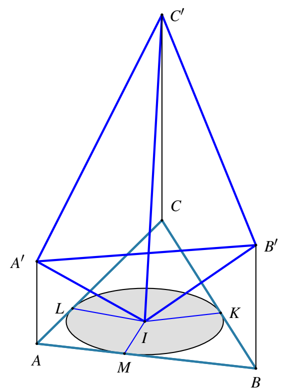
The Pavillet Tetrahedron is a unique orthocentric tetrahedron attached to a triangle called base triangle (vertices A, B, C). Its has numerous properties which can be used to prove classical triangle geometry theorems or, conversely, triangle geometry theorems can be used to prove some of its properties. It is built from the incircle ($\mathscr{C}_I$) of the base triangle (radius r). The incenter (I) is the apex of the tetrahedron, the other three vertices (A', B', C') form a triangle called the upper triangle.



The paper deals with five pencils, three pencils of spheres and two pencils of circles. They are

- The Euler coaxal pencil of spheres

  This is a very well known pencil of spheres of an orthocentric tetrahedron which is defined by the first and second twelve point spheres of this tetrahedron. Its axis is the Euler line of the tetrahedron.

- The Euler coaxal pencil of circles

  This pencil is also well known, it is a pencil of circles which is defined by the circumcirle and Euler circle of this triangle; its axis is the Euler line of this triangle.

- The pencil of spheres of the fourth altitude

  This pencil is described in a former paper, it is defined by the imaginary polar sphere of the Pavillet tetrahedron and the incircle sphere of the base triangle which is the sphere having the incircle of the triangle as great circle.

- The incircum pencils

  For this paper, we call incircum line the line going through the incenter and circumcenter of a triangle and we define the incircum pencil of circles of a triangle as the pencil of circles defined by its incircle and circumcircle. Its axis is the incircum line.

  Then we define the incircum pencil of spheres of a triangle as the coaxal pencil of spheres having the circles of the incircum pencil of circles as great circles with the same axis.

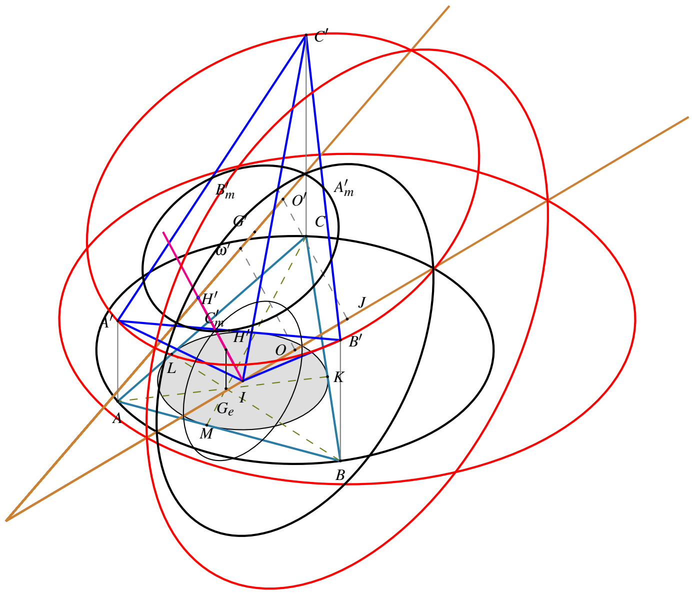


Figure 1: Trace of the incircum pencil of spheres on the upper plane

In the paper we use twice a classical theorem by Court about the intersection of a pencil of spheres by a plane. First it is fairly obvious that the intersection of the Euler coaxal pencil of sphere of the tetrahedron by the upper plane yields the Euler pencil of circle of the upper triangle. Then we show that the always imaginary basic circle of the pencil of spheres of the fourth altitude is the polar circle of the upper triangle which belongs to the Euler pencil of the upper triangle. We also use a result from ICGG2012 about the circumcircle sphere of the base triangle, the sphere of the incircum pencil which has the circumcircle of the base triangle as great circle, where we have shown that its trace on the upper plane is the Euler circle of the upper triangle. With these two results we show, with Desargues's theorem, that the trace of the incircum pencils of spheres of the base triangle on the upper plane is the Euler pencil of circles of the upper triangle.

We also find the radical axes and the sphere of the incircum pencil, called the support sphere, centered at the Bevan point which has the circumcircle of the upper triangle as trace on the upper plane.

It creates a remarkable correspondence between circles of the base triangle and circles of the upper triangle:

Table 1: Pencils correspondence

pencil	*circles*	*spheres*	*spheres*	*circles*
line of centers	incircum line	incircum line	Euler line Tetrahedron	Euler line Upper triangle
	incircle	incircle sphere	polar sphere	polar circle
	circumcircle	circumcircle sphere	1st twelve point sphere	Euler circle
	support circle	support sphere	circumsphere	circumcircle
	radical axis	radical plane	orthic plane	orthic axis

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