COAXAL PENCILS OF CIRCLES AND SPHERES IN THE PAVILLET TETRAHEDRON

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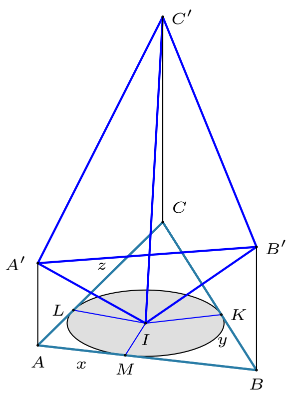
ABSTRACT: After a brief review of the properties of the Pavillet tetrahedron, we recall a theorem about the trace of a coaxal pencil of spheres on a plane. Then we use this theorem to show a remarkable correspondence between the circles of the base and those of the upper triangle of a Pavillet tetrahedron. We also give new proofs and new point of view of some properties of the Bevan point of a triangle using solid triangle geometry.

Keywords: Tetrahedron, orthocentric, coaxal pencil, circles, spheres polar

1 Introduction.

The orthocentric tetrahedron of a scalene triangle [12], named the Pavillet tetrahedron by Richard Guy [8, Ch. 5] and Gunther Weiss [16], is formed by drawing from the vertices A, B and C of a triangle ABC, on an horizontal plane, three vertical segments $AA' = AM = AL = x$, $BB' = BK = BM = y$, $CC' = CL = CK = z$, where KLM is the contact triangle of ABC. We denote I the incenter of ABC, r its in-radius, and consider the tetrahedron $IA'B'C'$.



This figure is rich in properties and various of them have been described in [12], [10] and [13].

2 Known properties

We first recall the notations and properties (with their reference) we will use in this paper (Fig. 1).

- The triangle ABC is called the *base triangle* and defines the *base plane* (horizontal).
- The incenter, I, is called the *apex* of the tetrahedron.
- The other three vertices (A', B', C') form a triangle called the *upper triangle* and define a plane called the *upper plane*.

As a standard notation, all points lying on the base plane will have (as much as possible) a standard triangle geometry notation, (e.g. O, circumcenter of the base triangle); all points lying on the upper plane will be denoted with primes (e.g. vertices A', B', C', circumcenter O', . . .); all internal points of the tetrahedron will be denoted with double primes (e.g. orthocenter H'', circumcenter O'', . . .). The three circles $\mathscr{C}_A$, $\mathscr{C}_B$, and $\mathscr{C}_C$ centered at A, B and C with radius x, y and z will be called the three mutually tangent circles (dashed circles, *externally* tangent, in Fig. 1).

This tetrahedron attached to the base triangle (vertices A, B, C) is *unique*. It is unique for two reasons, firstly this is an asymmetric tetrahedron, the apex I and the vertices A', B', C' have different properties (most of the types of tetrahedron studied in the literature have symmetric properties for all the vertices and faces); secondly, there is a one-to-one correspondence between the scalene base triangle and the *always acute* (cf. [12, Theorem 5.3]) upper triangle and the tetrahedron linking them is unique. The study of this tetrahedron links solid and triangle geometry, it offers an *edge view* of triangle geometry. It can be used two ways, using solid geometry to prove triangle geometry properties and theorems or using triangle geometry to prove properties of the tetrahedron.

- The tetrahedron $IA'B'C'$ is orthocentric (because $IB'^2 = 2y^2 + r^2$, while $A'C'^2 = 2x^2 + 2z^2$ so that the sum $IB'^2 + A'C'^2$ is independent of the sides) [12, p. 1].
- The lines $A'K$, $B'L$ and $C'M$ are three altitudes of the tetrahedron [12, Theorem 1.2].
- The altitude of the tetrahedron going through its apex I is called the *fourth altitude*.
- The orthogonal projection of the orthocenter H'' of the tetrahedron on the plane of ABC is G_e, the Gergonne point of ABC [12, Theorem 1.3].

We will also use:

- the five mutually orthogonal polar spheres of the orthocentric group A', B', C', I, H''. Represented on fig. 2 by their traces on the base plane, on the three vertical planes defined by $AA'BB'$ and on their siblings, they are:
  - The sphere centered at A' with radius $x\sqrt{2}$. It intersects the base plane along the corresponding mutually tangent circle. This sphere and its two siblings are called *vertex spheres* [12, Theorem 5.2].
  - The sphere centered at I with radius r. It is called the *incircle sphere* [12, Theorem 5.2].
  - An imaginary sphere centered at H'' with radius $n\sqrt{2}\mathrm{i}$ where $n = H''G_e$ [12, § 6].
- We define the *circumcircle sphere* of the base triangle as we defined the incircle sphere, i.e. the sphere which has the circumcircle of the base triangle as great circle. In [10, Corollary 3.1], we proved that the trace on the upper plane of the circumcircle sphere is the Euler circle of the upper triangle; a quicker proof is given in [13, Theorem 10].

  On figure 3, is drawn the circumcircle, the trace of the circumcircle sphere on the plane defined by the line OI and the fourth altitude; this plane, called the *Euler line plane*, is perpendicular to the upper plane and intersects it along the Euler line of the upper triangle [10, Corollary 3.2 , (a) and (b)]. The Euler circle of the upper triangle is centered at ω'.

We describe the coaxal pencils of circles and spheres we are going to use, noting that by application of the *always acute* property (cf. [12, Theorem 5.3]), these pencils are only of the non-intersecting type. We start with the most well-known.

3 The Euler pencil of an orthocentric tetrahedron

As described in [1], *any* orthocentric tetrahedron, generates a coaxal pencil of spheres which axis is the Euler line of the tetrahedron and contains the circumsphere (centered at O''), the first twelve point sphere (centered at the centroid G''), the second twelve point spheres (center denoted Γ'' on fig. 4) and the polar sphere of the tetrahedron (cf. [1, § 805]). The orthocentroidal

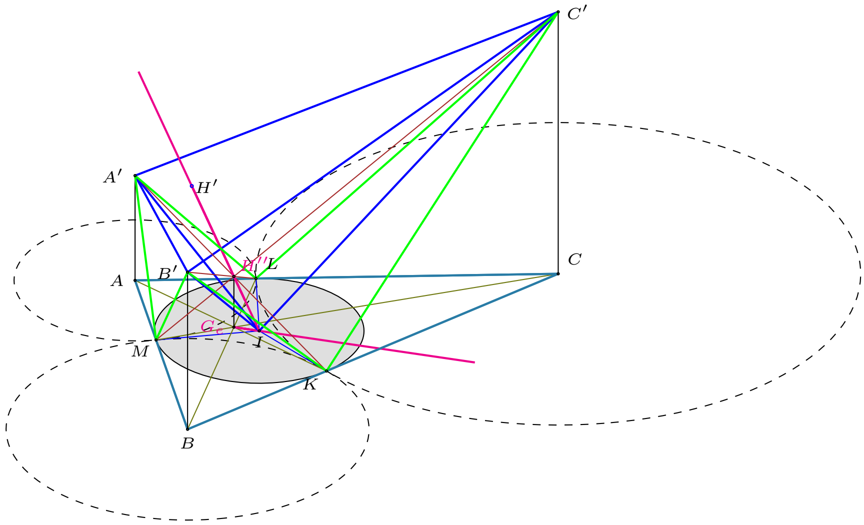


Figure 1: Orthocenter and Altitudes.

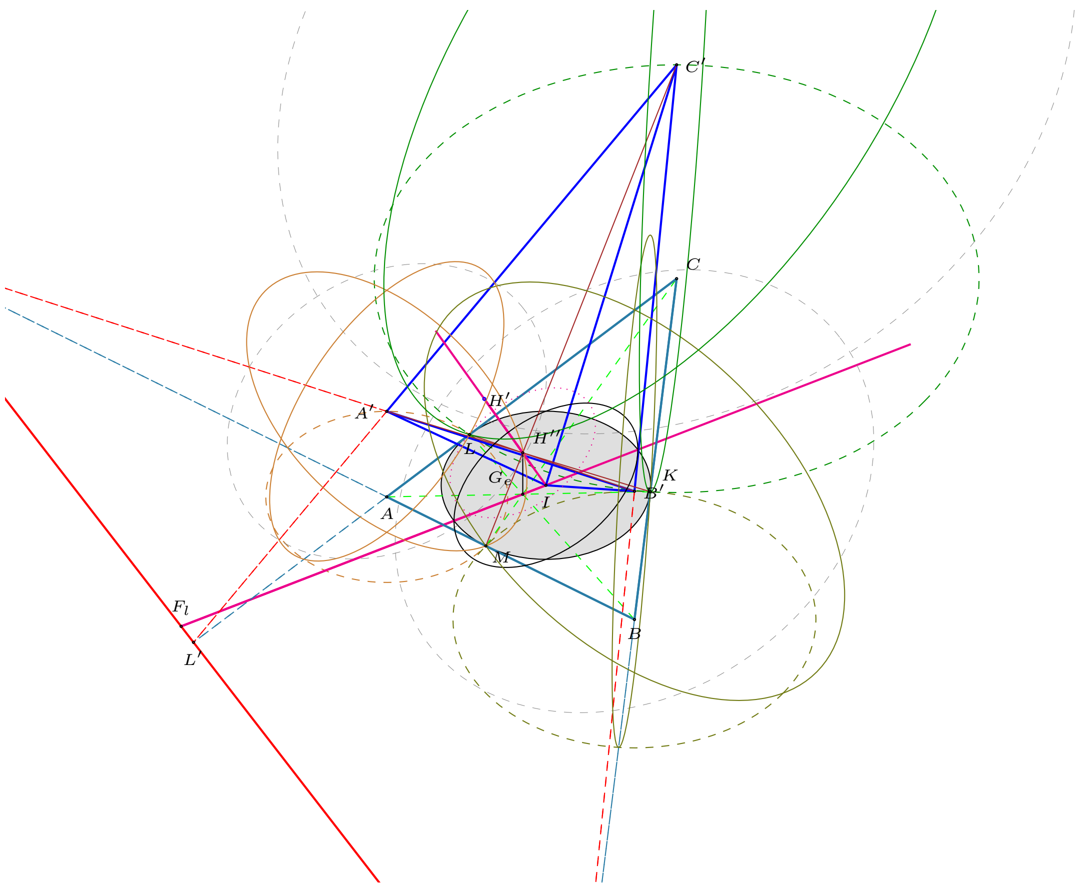


Figure 2: The Polar Spheres of the Orthocentric Group

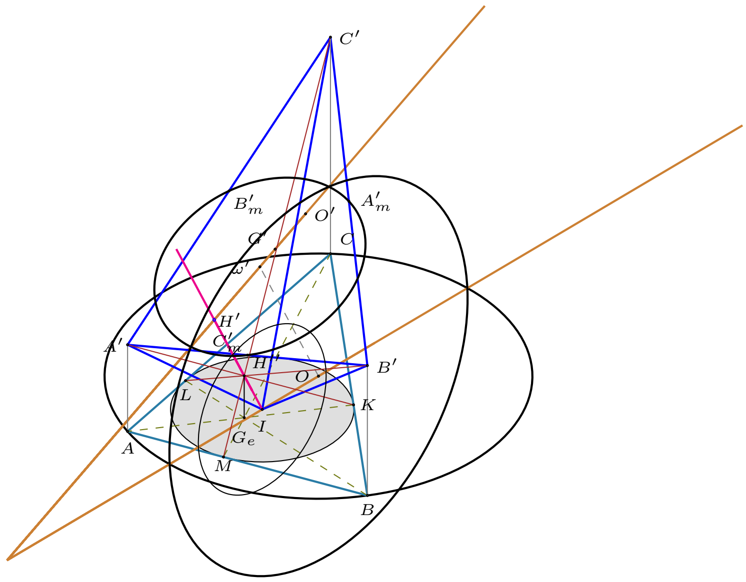


Figure 3: Trace of the Circumcircle Sphere on the Upper Plane

sphere, the sphere having the segment $H''G''$ joining the orthocenter to the centroid as diameter, also belongs to this pencil (cf. [1, § 807]). Finally the orthic plane of the tetrahedron (perpendicular to the Euler line at H_r'' on fig. 4) is the radical plane of this coaxal pencil (cf. [1, § 809]). We call this coaxal pencil the *Euler pencil of an orthocentric tetrahedron*.

4 Intersection of a coaxal pencil of spheres with a plane

It is clear that the intersection of a coaxal pencil of quadric with a plane yields a coaxal pencil of conic [14, § 13.15].

When we deal with spheres, a theorem by Court states [1, § 558]:

Theorem 4.1 *Any plane cuts a coaxal pencil of spheres along a coaxal pencil of circles.*

It comes with the following properties:

- The secant plane cuts the radical plane of the pencil of spheres along a line which is the radical axis of the system of circles.
- The line of centers of the coaxal pencil of circles is the orthogonal projection of the line of centers of the pencil of spheres upon the secant plane and therefore they are coplanar.

Remark 4.1 *A special case arises when the plane of intersection is the radical plane of the pencil because in this case we only get the basic circle, real or imaginary, of the pencil (provided we do not consider the radical plane as a sphere of the pencil).*

5 The Euler pencil of a triangle.

From Theorem 4.1, the intersection of the Euler pencil of an orthocentric tetrahedron with one of its face therefore yields a coaxal pencil of circles. Now, the orthogonal projection of the Euler line of an orthocentric tetrahedron on one of its face is the Euler line of the face triangle: indeed

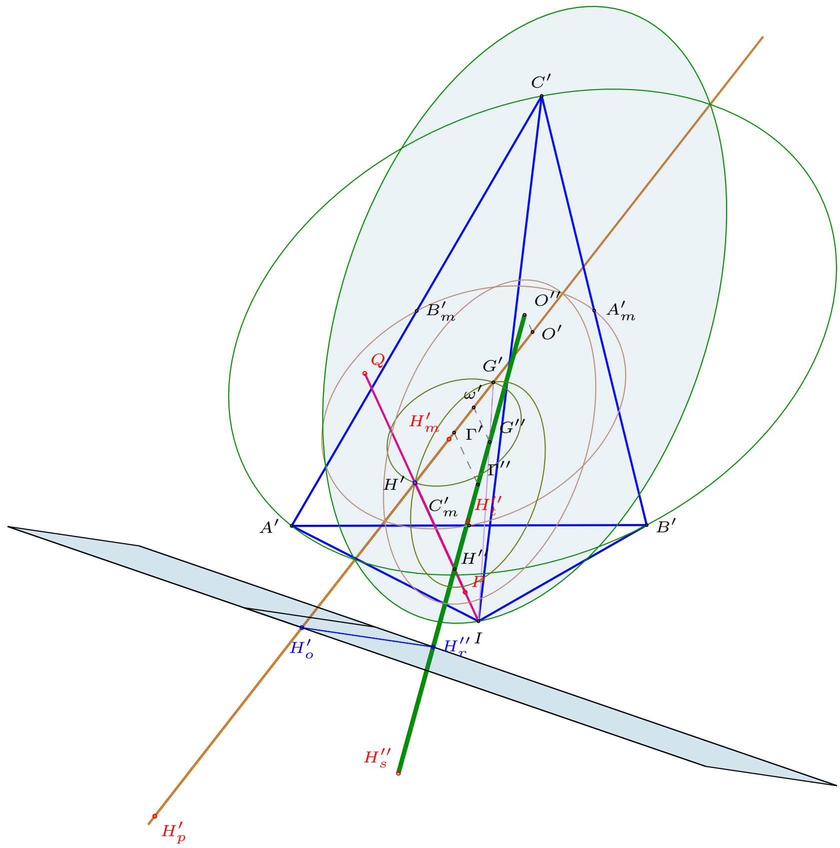


Figure 4: Cross-section of the Euler pencil of an orthocentric tetrahedron along both the Euler line plane and the upper plane.

orthocenter and circumcenter of the tetrahedron become orthocenter and circumcenter of the face triangle by orthogonal projections. Moreover the trace of the circumsphere on the face is the circumcircle of the face triangle while the trace of the first twelve point sphere is its Euler circle.

But we also know that, in a triangle, it exists a coaxal pencil of circles defined by the circumcircle and the Euler circle and from Desargue's theorem a coaxal pencil of quartics surfaces or conics curves will intersect a straight line at points which are conjugate pairs of an involution [15, Theorem 23, p. 130]. So both pencils create an involution on the Euler line of the face triangle but these two involutions have two conjugate pairs in common, one given by the intersection with the circumcircle and circumsphere and one given by the intersection with the first twelve point sphere and the Euler circle. Hence the pencils of spheres and the pencil of circles determine the same involution on the Euler line and therefore the pencil of circles determined by the trace of the pencil of spheres and the pencil determined by the circumcircle and Euler circle are identical.

We call it the *Euler pencil of the triangle* (in our case, the Euler pencil of the upper triangle). The circumsphere (centered at O'') yields the circumcircle of the triangle (centered at O'), the first twelve point sphere (centered at G'') yields the Euler circle of this triangle (centered at ω'). The second twelve point sphere is also a sphere of the Euler pencil of the tetrahedron. Its center lies at $1/3$ of the distance from the orthocenter to the circumcenter in the direction of the circumcenter. It goes through the centroids and the orthocenters of the faces (cf. [1, § 800]). Therefore, the orthocentroidal circle of the upper triangle is not the trace of the orthocentroidal sphere of the tetrahedron but the trace of the second twelve point sphere on the upper plane.

This is an improbable solution to an exercise proposed in [2, Section VII § 110, p. 160].

Most importantly, for an acute tetrahedron, using a definition from [5, § 6.2])

Theorem 5.1 *The trace of the imaginary polar sphere of the tetrahedron (centered at H'') on the face triangle is the imaginary polar circle (centered at H') of this triangle.*

And, because the intersection of the orthic plane with the upper plane gives the orthic axis of the upper triangle (cf. [1, § 810]), we have a solid geometry proof that the orthic axis of a triangle is the radical axis of the pencil of circles of the Euler line of any triangle.

6 The pencil of spheres of the fourth altitude of the Pavillet tetrahedron

In [12, § 5] we have shown that the following spheres (Fig. 5) were coaxal:

- The incircle sphere, center I, radius r, the sphere having the incircle as great circle.
- The *inner and outer tritangent spheres* whose intersections with the base plane gives the inner and outer Soddy circles of ABC.
- The imaginary polar sphere of the tetrahedron, radius $n\sqrt{2}\mathrm{i}$, center H''.
- The upper plane, radius $+\infty$, radical plane of the pencil.
- Two null spheres centered at P and Q where P and Q are, either the two points of intersection of the polar spheres of $IA'B'C'$ centered at A', B', C' or, the two points of intersection of the three *edge spheres* (see [8, Ch. 2, p. 36]), i.e the spheres having A'_m, B'_m, C'_m, the mid points of $B'C'$, $C'A'$ and $A'B'$, as centers and $B'C'$ and so on as diameters.

We call this pencil of spheres *the pencil of the fourth altitude*.

This pencil having limiting points is of the non-intersecting type. From Remark 4.1 and Theorem 5.1, we get the following corollary:

Corollary 6.1 *In a Pavillet tetrahedron, the basic circle of the pencil of the fourth altitude is the imaginary polar circle of the upper triangle.*

7 The incircum pencils

Definition 7.1 *We call* incircum pencil of circles of a triangle *the coaxal pencil of circles formed by the incircle and circumcircle. Its line of centers will be called the* incircum line.

In the normalization of the Encyclopedia of Triangle Centers this line, OI, is called $Line\,1,3$.

We note that by its very definition, this pencil is of the non-intersecting type.

Similarly, we define

Definition 7.2 *We call* incircum pencil of spheres *of the base triangle the coaxal pencil of spheres which have the circles of the incircum pencil of circles as great circles.*

Now we know from 6.1 that the trace of the incircle sphere on the upper plane is the polar circle from the upper triangle and we already recalled that the trace of the circumcircle sphere of the base triangle on the plane of the upper triangle

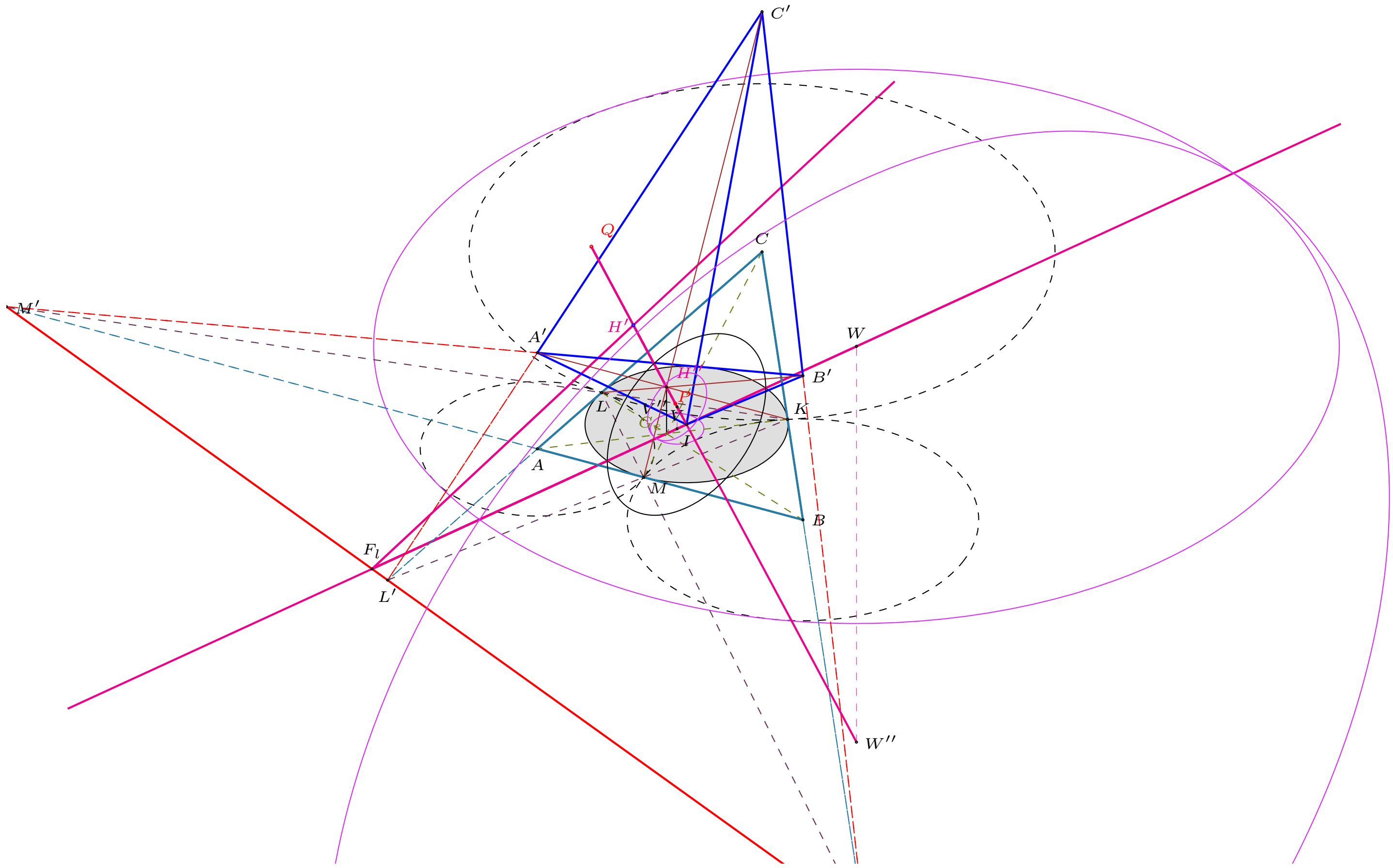


Figure 5: The coaxal pencil of spheres of the fourth altitude.

was the Euler circle of this triangle. Therefore, using again Desargues's Theorem and the fact that an involution is completely determined by two sets of conjugate points we conclude that the involutions determined by the incircum pencil of spheres of the base triangle and the Euler pencil of spheres of the Pavillet Tetrahedron on the Euler line of the upper triangle are identical.

We have proved

Theorem 7.1 *The trace of the incircum pencil of spheres of the base triangle on the upper plane is the Euler pencil of circles of the upper triangle.*

Combining this theorem with the properties of the radical plane coming from Theorem 4.1, we have also proved

Corollary 7.1 *The orthogonal projection of the orthic axis of the upper triangle on the base plane is the radical axis of the incircum pencil of circles of the base triangle.*

8 The support sphere

We have one question left, we know the circles of the base triangle corresponding to the polar circle and the Euler circle of the upper triangle (incircle and circumcircle). What could be the circle corresponding to the circumcircle of the upper triangle?

Though the relationship is not obvious here, to be consistent with work done in [9, Definition 21], we define

Definition 8.1 (Support sphere) *We call support sphere, the sphere of the incircum pencil of spheres which trace on the upper plane is the circumcircle of the upper triangle.*

The support circle will be the corresponding circle of the incircum pencil of circles[1]. We call J the center of the support sphere and circle, it lies on the incircum line. If ω' is the center of the Euler circle of the upper triangle and O' its circumcenter, we have by property of the Euler line $\omega' H' = \omega' O'$ and the three points H', ω', O' are orthogonal projections of I, O by the former properties and J by definition. Hence they all lie on the incircumline and we have $OI = OJ$ so that J is the Bevan point of the base triangle (X(40) in the Encyclopedia of Triangle Centers).

To get to the circle itself, we can use a theorem by Coolidge, [4, Theorem 206, p. 107]:

Theorem 8.1 (Theorem 206) *If three circles have non-collinear centres, and their radical centre lies outside of them, then the circle through their three centres is the radical circle of their common orthogonal circle and that circle, when it exists, which cuts each of the three in a pair of diametrically opposite points. ...*

Here the three circles are the three mutually tangent circles, their radical center is the incenter, lying outside of them. The circle through their three centers is the circumcircle which is the radical circle of the incircle (common orthogonal circle to the three mutually tangent circles) and that circle which has to be centered at the Bevan point (the center of the radical circle is halfway between both centers) which cuts each of the three mutually tangent circles in a pair of diametrically opposite points. Therefore, incircle, circumcircle and the great circle of the support sphere centered at J, the Bevan point, cutting diametrically the three mutually tangent circles are coaxal.

[1]This is *not* the Bevan circle

To get a different insight about the Pavillet tetrahedron, we give another proof, where we visualize imaginary circles in the horizontal plane using null spheres (called foci) in space whose intersections with the plane of the circle are those imaginary circles. This old idea is taken from the works of Chasles [3, § 790-798, p. 506] and Darboux [6, § XV - p. 369][2].

The intersection of a sphere $\mathscr{S}$, center O, with a plane Π at a distance h of $\mathscr{S}$ is a circle which center is the foot H of the perpendicular to the plane going through the center of the sphere. If R is the radius of the sphere and h the distance OH, the square of the radius of the circle of intersection is given by

$$R^2 - h^2. \tag{8.1}$$

This formula applies for R real, pure imaginary and also if $R = 0$. Therefore using null sphere, the modulus of the radius of the imaginary circle is the distance of the focus from the plane.

So because, by definition, A', B', C' lie on the support sphere, the null spheres centered at A', B', C' are orthogonal to the support sphere. A cross-section of two orthogonal spheres by a plane will yields two orthogonal circles provided the plane is going through at least one of its two centers. Hence the cross-section of the four spheres by the base plane will yields three circles centered at A, B and C, clearly imaginary, orthogonal to the support circle whose foci are exactly A', B', C'.

Again, by definition of A', the distance $A'A$ is its altitude. But x is also the radius of the mutually tangent circle centered at A so that this real circle is cut diametrically by the support circle.

We have proved,

Theorem 8.2 *The circle centered at the Bevan point which cuts diametrically the three mutually*

[2]There is also an algebraic definition of the foci given in [4, Ch. XII, § 1], but it does not allow for easy synthetic proofs

tangent circles belongs to the incircum pencil of circles, the trace on the upper plane of the corresponding sphere is the circumcircle of the upper triangle.

In fact we can prove all properties of the Bevan point using these three imaginary circles and the Pavillet tetrahedron.

9 Solid triangle geometry

To each of the three mutually *externally* tangent circles such as $\mathscr{C}_A$, centered at A with radius $AL = AM = x$ we associate a concentric (and orthogonal!) imaginary circle denoted ${}^{\perp}\mathscr{C}_A$, with radius $\mathrm{i}\,x$. Clearly the *real* foci of the three circles ${}^{\perp}\mathscr{C}_A$, ${}^{\perp}\mathscr{C}_B$ and ${}^{\perp}\mathscr{C}_C$ are the six vertices A', B', C', $\bar{A}'$, $\bar{B}'$, $\bar{C}'$ of the conjugate Pavillet tetrahedra [11][3].

To get the radical center of these three imaginary circles means finding a point *lying on the horizontal plane* equidistant of the three (or six by symmetry) vertices A', B', C' (Fig. 6). This point has to lie on the perpendicular to the upper plane at the circumcenter of the upper triangle (intersection of the mediator planes of $A'B'$, $B'C'$ and $C'A'$).

Therefore the Bevan point is the radical center of the three imaginary circles with radius $\mathrm{i}\,x$, $\mathrm{i}\,y$ and $\mathrm{i}\,z$ centered at A, B and C because the distances $JA' = \sqrt{JO'^2 + O'A'^2}$, $JB' = \sqrt{JO'^2 + O'B'^2} = \sqrt{JO'^2 + O'A'^2}$ and $JC' \ldots$ are equal.

Because the base triangle is arbitrary, we have proved that

Theorem 9.1 *In a triangle, the radical center of the three circles associate to the three mutually tangent circles is the Bevan point.*

[3] Note that they are not mutually tangent to one another.

Now, on the base plane, we can use these associate circles to give some new proofs of well known properties.

The radical axis of two of the mutually tangent circles (e.g. $\mathscr{C}_A$ and $\mathscr{C}_B$), is perpendicular to the line of centers, i.e. the side of the triangle (e.g. AB). It intersects the side at the corresponding vertex of the contact triangle (e.g. M) and goes through the radical center of these three circles, the incenter I. Similarly the radical axis of two of the three associate circles (e.g. ${}^{\perp}\mathscr{C}_A$ and ${}^{\perp}\mathscr{C}_B$) is perpendicular to the same side (e.g. AB).

Now (fig. 7), the orthogonal projection of the circumcenter on the side of the triangle being the midpoint of this side and the Bevan point being the symmetric of the incenter with respect to the circumcenter, we have

Corollary 9.1 *The intersection of the radical axis of two associate circles with their line of centers is isotomic of the corresponding vertex of the contact triangle.*

Of course, the vertex of the contact triangle is not only the intersection of the radical axis of two of the mutually tangent circles with their line of centers but is also the foot of the orthogonal projection of the incenter on the corresponding side while its isotomic point is the foot of the orthogonal projection of the corresponding excenter on the same side, we have proved

Theorem 9.2 *The radical axis of two of the associate imaginary circles goes through the corresponding excenter.*

Otherwise we could have seen that the corresponding excenter I_C is such that its power about the associate circle is ${}^{\perp}\mathscr{C}_A$ is $I_C M_x^2 + M_y A^2 + AA'^2 = r_c^2 + y^2 + x^2$ while its power about ${}^{\perp}\mathscr{C}_B$ is $r_c^2 + x^2 + y^2$.

And so, we see that the perpendiculars drawn from the excenters on the sides of the triangle are

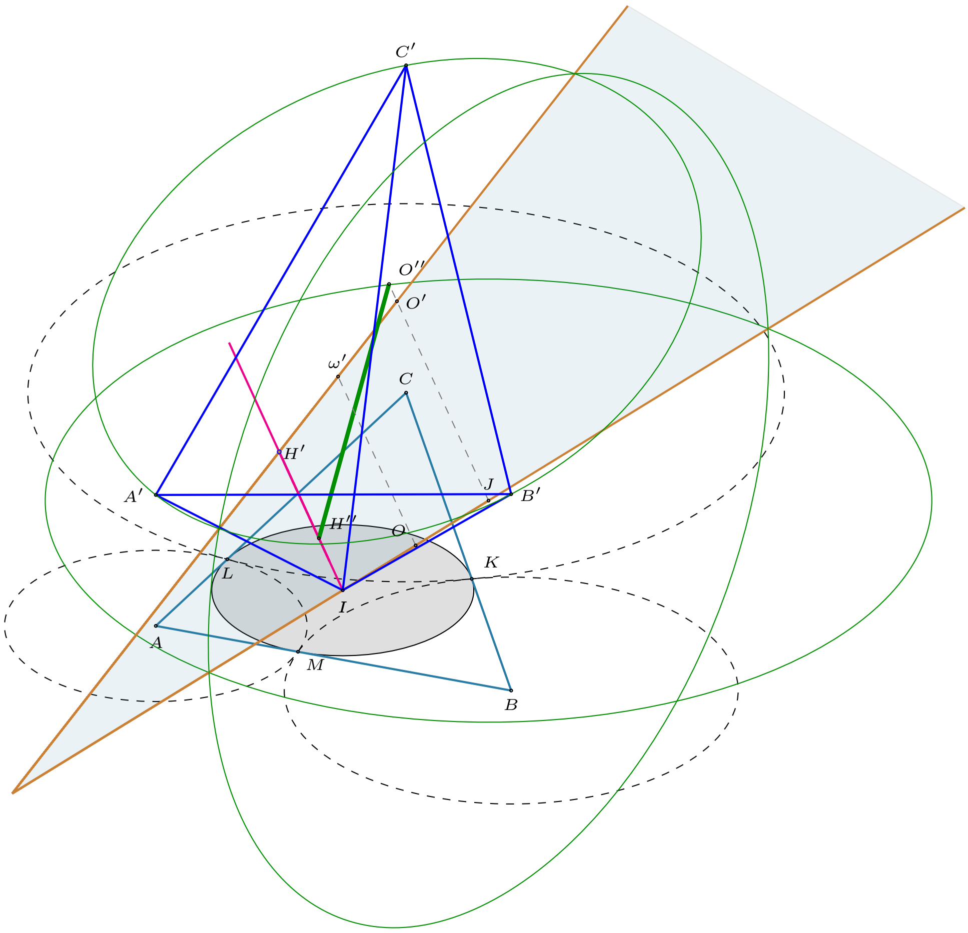


Figure 6: The Support Sphere Centered at the Bevan Point

concurrent at the Bevan point. In fact the proof given here holds only for the internal sides but using Conway's extraversion and Richard Guy generalization of the Pavillet tetrahedron to the excircles the result could be extended [8, § 5.3, p. 68].

10 Conclusions.

In this limited paper, we have shown an interesting correspondence (table 1) between two important coaxal pencil of circles of triangles linked by a mere orthogonal projection using the cross-section of coaxal pencil of spheres by a plane. However it should be noted that using the same technique of intersection by a plane we got broader results using coaxal nets of spheres and their conjugate pencil. For example, the intersection of the coaxal net of intersecting spheres defined by the three vertex spheres centered at A', B', C' and its conjugate pencil (the pencil of the fourth altitude) by the base plane yields the coaxal system formed by the incircle and two Soddy circles of a triangle with the Soddy line as line of centers, the Gergonne line as radical axis and its orthogonal coaxal system (cf. [8, Theorem p. 60]).

Finally, there are other known circles belonging to the incircum pencil, one of them is the circle going through the Gergonne point of the triangle [7, § 34], finding the properties of the corre-

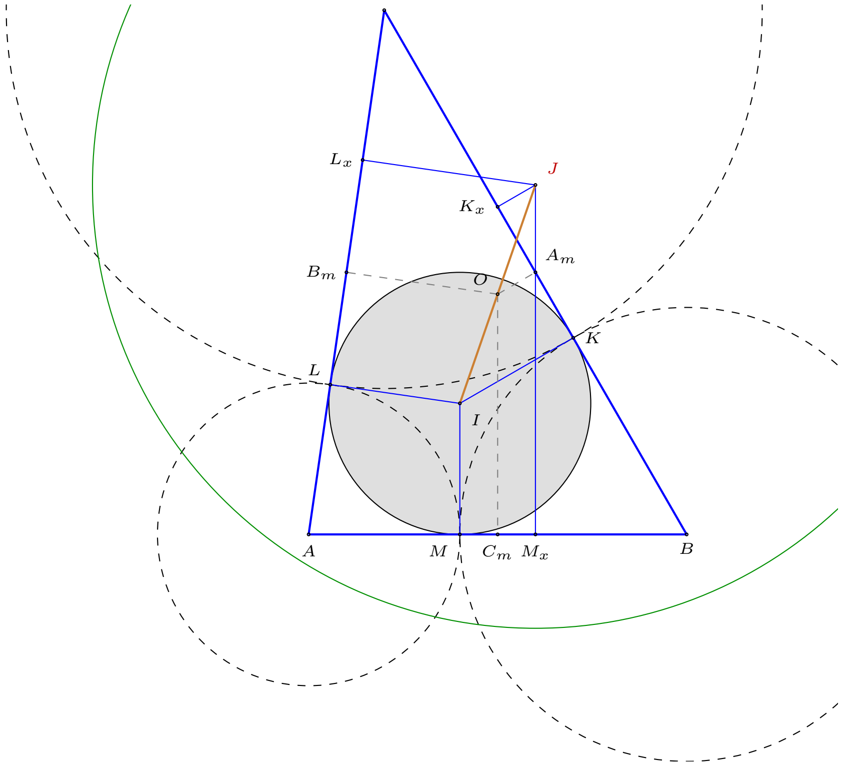


Figure 7: Property of the Bevan Point

sponding circle of the upper triangle is another possible extension of this work (G_e lying inside the incircle, it is always an imaginary circle but then its foci will be real).

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Table 1: Pencils correspondence

pencil	*circles*	*spheres*	*spheres*	*circles*
line of centers	incircum line	incircum line	Euler line Tetrahedron	Euler line Upper triangle
	incircle	incircle sphere	polar sphere	polar circle
	circumcircle	circumcircle sphere	1st twelve point sphere	Euler circle
	support circle	support sphere	circumsphere	circumcircle
	radical axis	radical plane	orthic plane	orthic axis

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