THE ORTHOCENTRIC TETRAHEDRON OF A TRIANGLE
NEW PROPERTIES AND INVERSE PROBLEM

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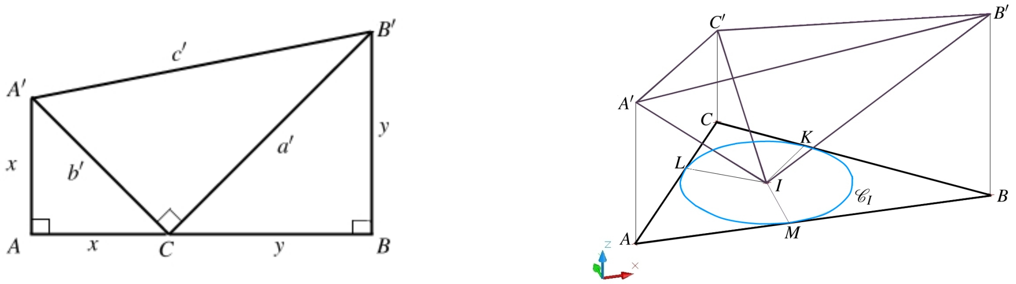
“It is popularly supposed (not least by some mathematical educators) that there are no new theorems in elementary geometry, but . . . ” (J.V. Armitage [1])

1 The Orthocentric Tetrahedron of a Triangle

To the best of our knowledge, the orthocentric tetrahedron of a triangle is the first three dimensional object directly associated with a triangle.

Consider the following segment AB in $\mathbb{R}$ with a point C in between (fig. 1- a). To go from $1D$ to $2D$, drawing $AA' = AC$ and $BB' = BC$, we get a triangle $A'CB'$ right in C. The segment ABC may be interpreted as a flat triangle with the zero radius circle centered at C as its incircle. One may use a similar idea to now go from 2 to 3 dimensions. Starting from a triangle ABC on the horizontal plane (fig. 1- b), called the base triangle, with associated incenter I, incircle $\mathscr{C}_I$ and contact triangle KLM, we draw the vertical segments $AA' = AM = AL$, $BB' = BK = BM$ and $CC' = CL = CK$, $A'B'C'$ is called the upper triangle. The orthocentric tetrahedron of ABC is the tetrahedron $A'B'C'I$.



(a) from $\mathbb{R}$ to $\mathbb{R}^2$ (b) from $\mathbb{R}^2$ to $\mathbb{R}^3$

Figure 1: Building the Tetrahedron.

Its main properties are described in [4], others in [2]. The defining property of this tetrahedron is that the orthogonal projection of its orthocenter on the plane of its base triangle is the Gergonne point of this triangle.

The vertices do not play an equal role in this tetrahedron and the incenter of the base triangle is called the apex of the tetrahedron. The altitude and median going through the apex are called the fourth altitude and fourth median. With the tetrahedron come four spheres: three vertex spheres (centered on A', with radius $A'M$) and the incircle sphere, the sphere having $\mathscr{C}_I$ for diametral circle (fig. 1- b). The following properties of [4] will be used:

- The four spheres are mutually orthogonal polar spheres of the orthocentric group defined by $A'B'C'I$.

- The trace of the plane of the upper triangle on the base plane is the Gergonne line of the base triangle.

- The orthocentric tetrahedron of a triangle is always *acute*.

The paper develops a number of new properties for this object, they add further connections between a triangle and its orthocentric tetrahedron.

2 New Properties of this Tetrahedron

The traces on the upper plane of the three vertex spheres are a set of three mutually orthogonal circles having the orthocenter of the upper triangle as radical center. We consider three concentric spheres having their radius reduced in a ratio $\frac{1}{\sqrt{2}}$, to get a set of reduced spheres. We also consider a new sphere, the circumcircle sphere as the sphere having the circumcircle of the base triangle for diametral circle. We show that

- The radical axis of the three reduced spheres is a parallel to the fourth altitude going through the circumcenter of the base triangle and the Euler center of the upper triangle.

- The centroid of the tetrahedron belongs to this radical axis.

- The first twelve point sphere of the tetrahedron and the circumcircle sphere have the same trace on the upper plane, the Euler circle of the upper triangle.

- The orthogonal projection of the Euler line of the contact triangle KLM (fig. 1- b) on the upper plane is the Euler line of the upper triangle $A'B'C'$.

- The orthogonal projection of the centroid of the tetrahedron lies on the Housel line of the base triangle ABC and is the harmonic conjugate of the incenter of the medial triangle about the centroid and the incenter of ABC.

3 Geometric Solution of the Inverse Problem

Finding $A'B'C'$ from ABC and therefore the construction of $A'B'C'I$ starting from ABC is easy. We define the inverse problem the following way: given an *acute* triangle $A'B'C'$ can we find an orthocentric tetrahedron $A'B'C'I$ and a base triangle ABC such that $A'B'C'I$ is the orthocentric tetrahedron of ABC?

Using an ad-hoc procedure to solve the inverse problem can be graphically intensive. With the new properties linked to the Euler circle, we give an elegant solution and a fast procedure to solve this inverse problem. Solid Geometry has evolved, it is now more experimental and most of this research is done with software, so the paper will show additional steps to solve it with a CAD software as an application of [3].

4 Conclusion

Associating tri-dimensional objects to a triangle is a fruitful operation, we could call this "'*Solid Triangle Geometry*"'. In [4] the research was centered on the fourth altitude. In this paper, we prove new properties linked to the Euler lines and the 4^{th} median. We show how connected the triangle is with its orthocentric tetrahedron. Using some of these new properties we derive a construction procedure for the inverse problem and show that, up to a symmetry, we have a single solution.

Finally, though Euclidean Solid Geometry is a very mature discipline, this paper illustrates that there are potentially a lot of beautiful properties left to discover and the use of software may facilitate this process.

References

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