THE ORTHOCENTRIC TETRAHEDRON OF A TRIANGLE NEW PROPERTIES AND INVERSE PROBLEM

Axel PAVILLET
axel.pavillet@polytechnique.org

ABSTRACT: To the best of our knowledge, the orthocentric tetrahedron of a triangle is the first three dimensional object directly associated with a triangle. Considering a triangle ABC on the horizontal plane, called the base triangle, its incenter I, incircle $\mathscr{C}_I$ and KLM its contact triangle, we draw the vertical segments $AA' = AM$, $BB' = BK$, $CC' = CL$. The orthocentric tetrahedron of ABC is the tetrahedron $A'B'C'I$. Its main properties and the links with classical theorems of triangle geometry are described in [7], others are described in [4]. The defining property of this tetrahedron is that the orthogonal projection of its orthocenter on the plane of its base triangle is the Gergonne point of ABC. Finding $A'B'C'$, called the upper triangle, from ABC and therefore the construction of $A'B'C'I$ starting from ABC is easy, theoretically and practically with a CAD software.
The paper develops a number of new properties for this object, they add further connections between the triangle and its tetrahedron. The inverse problem is well defined if given a triangle $A'B'C'$ we can find an orthocentric tetrahedron $A'B'C'I$ and a base triangle ABC such that $A'B'C'I$ is the orthocentric tetrahedron of ABC. Using only the initial properties to solve the inverse problem can be graphically intensive. In this paper, using new properties linked to the first twelve point sphere, we give an elegant solution and a fast procedure to solve this inverse problem. Solid Geometry has evolved, it is now more experimental and most of this research is done with software, so the paper will show additional steps to solve it with a CAD software as an application of [6].

Keywords: Euclidean, Plane, Solid, Triangle, Geometry, Orthocentric, Tetrahedron, CAD software

1 Known Properties

We use the notations of figure 1. With the orthocentric tetrahedron comes the five polar spheres of the orthocentric group [1, §833 p. 274], they are: three vertex spheres: centered on A', with radius $A'M = x\sqrt{2}$, the incircle sphere, radius r, the sphere having $\mathscr{C}_I$ for diametral circle and an imaginary sphere centered on the orthocenter, all mutually orthogonal. The perpendicular to the plane of the upper triangle at the orthocenter going through I, called the apex of the tetrahedron, is called the fourth altitude .

The following properties will be used, they all come from [7]:

a) The length m of the bimedian of the tetrahedron is given by

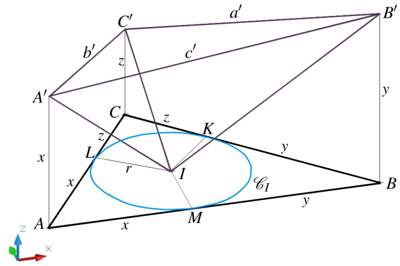

Figure 1: notations

$$4m^2 = A'I^2 + B'C'^2 = 2(x^2+y^2+z^2)+r^2. \quad (1)$$

b) The radical axis of the three vertex spheres is the fourth altitude, IH' (fig. 2), of the tetrahedron and the three spheres belongs to an intersecting coaxal net of spheres. The two common points of the net are denoted P and Q.

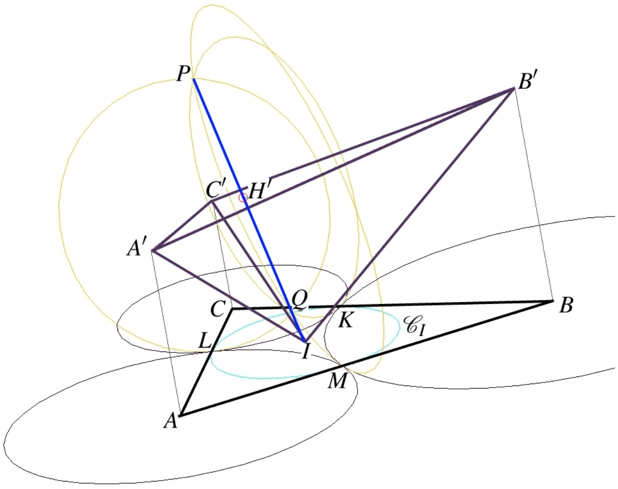

Figure 2: Three vertex spheres and their radical axis

c) The trace of the upper plane on the base plane is the Gergonne line, Γ, of the base triangle (fig. 3).

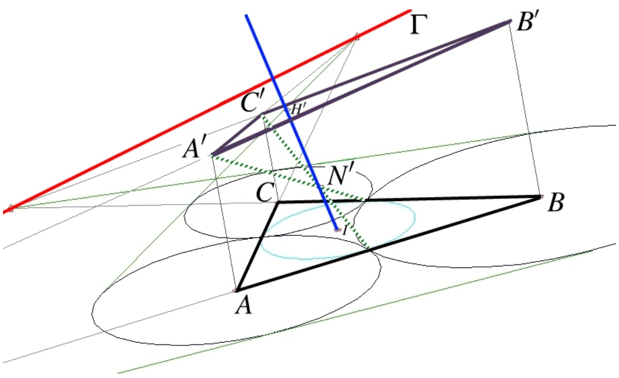

Figure 3: Intersection upper plane base plane, Gergonne line of ABC

d) Whatever the base triangle, the orthocentric tetrahedron of a triangle is always *acute*. It means, equivalently, either that the polar sphere of the tetrahedron is imaginary, or that its orthocenter is always within the tetrahedron or that all faces are acute triangles.

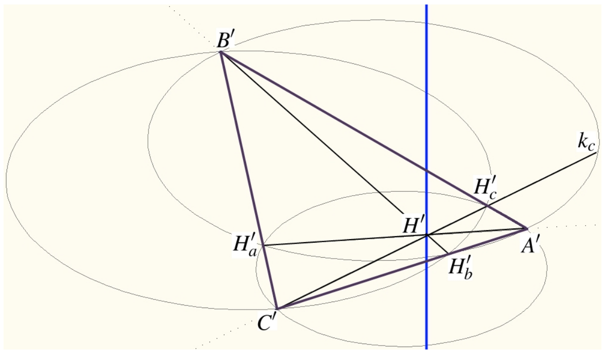

Figure 4: Fourth altitude

2 Initial construction

Using only the initial properties of the tetrahedron, we show, summarily, how, given a triangle $A'B'C'$, to construct a base triangle ABC (incenter I) so that its orthocentric tetrahedron is $IA'B'C'$, i.e. $IA'B'C'$ has the triangle $A'B'C'$ as upper face. The proof given here is constructive and practical: it is done in such a way that it can be done with the usual commands of a CAD software, we do it as a direct application of [6].

Interestingly, we can easily draw the orthocenter H' of $A'B'C'$, and the perpendicular to the plane $A'B'C'$ going through H'. This perpendicular is the fourth altitude of the tetrahedron we are looking for (fig. 4) therefore the apex I lies on this altitude but we will not use this property.

1. Draw three circles with the sides of $A'B'C'$ as diameter. The circle with diameter $A'B'$ goes through the feet H'_a and H'_b of the altitudes through A' and B' and intersects $C'H'_c$ at k_c (fig. 4).

2. From vertex A' of $A'B'C'$ as center, draw a circle, $\mathscr{C}_{A'}$, going through k_c. Cycling through the vertices draw the other two similar circles. These three circles are orthogonal (the angle $\widehat{A'k_cB'}$ is right) and their radical center is H', the orthocenter of the triangle.

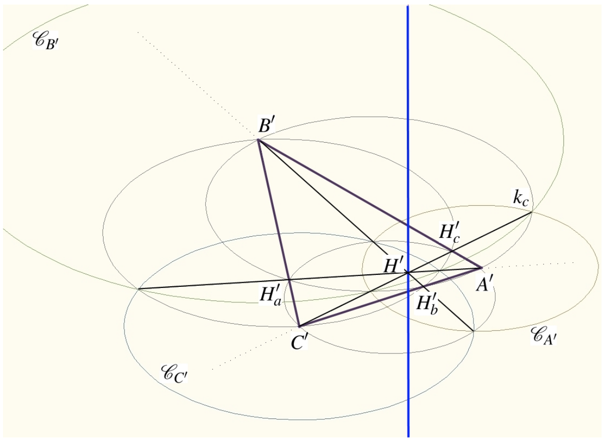

Figure 5: Traces on the upper plane $A'B'C'$ of the three orthogonal spheres

Therefore $\mathscr{C}_{A'}$, $\mathscr{C}_{B'}$ and $\mathscr{C}_{C'}$, are the great circles of the three orthogonal spheres centered on the upper vertices of the tetrahedron we are looking for (fig. 5). We have constructed a cross-section of the coaxal net of vertex spheres through the plane of centers of the net.

Remark 2.1 *If the triangle is obtuse, then H' is outside the triangle $A'B'C'$ and therefore one of the altitudes does not intersect the circle having the opposite side for diameter. There is no real solution.*

3. Draw the three corresponding orthogonal spheres $\mathscr{S}_{A'}$, $\mathscr{S}_{B'}$ and $\mathscr{S}_{C'}$ (fig. 6).
4. Draw the three circles of intersection of these three spheres $\mathscr{C}_{A'B'}$, $\mathscr{C}_{B'C'}$ and $\mathscr{C}_{C'A'}$, let P, Q be the common points of the net, (fig. 7), as mentioned, PQ is the fourth altitude.
5. Drawing the plane tangent to these three, non coplanar circles, will give us three points of contact which determine the plane of the base triangle because they are the contact triangle KLM of the base triangle ABC (fig 2).

   Now, even using software, the problem is

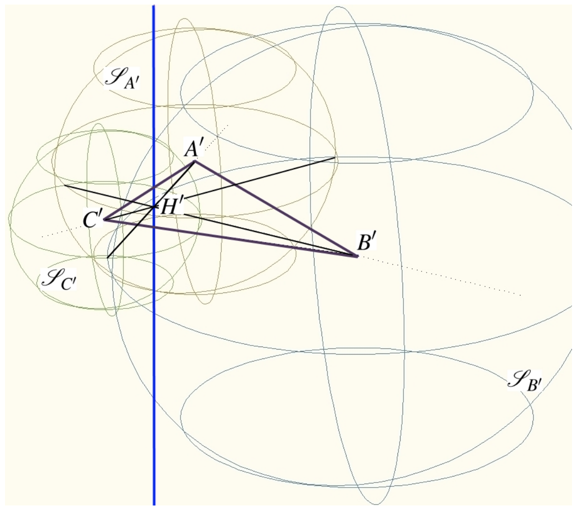

Figure 6: The three orthogonal spheres

to effectively find these three points. It can be done by inversion about P (or Q) with power PQ^2. The three circles of intersection will give us a trirectangular trihedron Q_x, Q_y, Q_z, (axis Δ) and the plane of the triangle will be transformed in a sphere, going through P, tangent to the edges of the trihedron (fig. 8). So we can draw this sphere, find T_x, T_y and T_z its points of contact with the edges of the trihedron and inverse back to get K, L and M. This is a fairly lengthy construction.

6. These three inversed points determine the plane of the base triangle because they are the contact triangle KLM of the base triangle ABC. The apex I is the circumcenter of

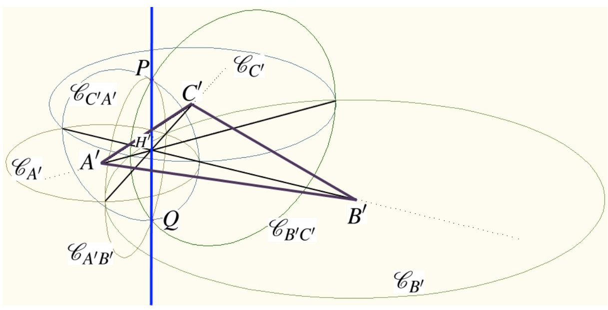

Figure 7: Construction of the three circles of intersection

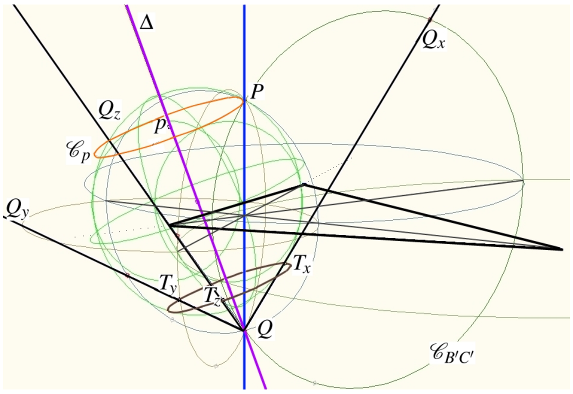

Figure 8: The sphere tangent to the trihedron

the triangle KLM found the step before.

7. This point is also the incenter of the triangle which has this tetrahedron as orthocentric tetrahedron. We get the three vertices A,B,C of the base triangle either drawing the perpendiculars to the plane KLM going through A', B', C' and intersecting them with the plane K, L, M or drawing the tangential triangle of KLM, i.e. taking the intersection of the three tangents at K,L and M to the circumcircle of the triangle KLM.

Note that, because we have two choices for the center of inversion, we have two identical solutions symmetrical about the plane $A'B'C'$.

3 Properties linked to the circumcircle

To simplify this construction, we look for new properties linked to the circumcircle of the base triangle.

Once we construct the three great circles $\mathscr{C}_{A'}, \mathscr{C}_{B'}, \mathscr{C}_{C'}$ of the the three vertex spheres (cf. § 2), we know their radii $x\sqrt{2}$, etc. Therefore, we can also draw three concentric circles $\mathscr{C}'_{A'}, \ldots$ of radius x, ..., they are the great circles of three spheres $\mathscr{S}'_{A'}$, ...centered on the vertices A', B', C' which are tangent to the base plane at A, B, C. We call them *reduced circles* and *reduced spheres*.

On the one hand, the d'Alembert line [3, p. 1261] of the three external centers of similitudes of these three circles is easy to find. Because the vertex spheres and the reduced spheres have their radii in the same ratio, $\sqrt{2}$, this line is also the line joining the external centers of similitude of the three vertex spheres, therefore it is also Γ, the Gergonne line of the base triangle ABC we are looking for.

On the other hand, the radical axis of these three reduced spheres is also perpendicular to the plane $A'B'C'$ and therefore parallel to the fourth altitude of the tetrahedron. This radical axis is going through O, the circumcenter of the base triangle, because the power of O about each of these three reduced spheres is R^2, where R is the radius of the circumcircle.

We have already defined the incircle sphere (§1), we similarly define the *circumcircle sphere* as the sphere having the circumcircle of the base triangle for diametral circle. It is orthogonal to the three reduced spheres because again the power of the circumcenter about the three spheres is equal to R^2.

The radical center of the three reduced spheres and the circumcircle sphere is also on this parallel to the fourth altitude and the traces on the plane of the base triangle of the three radical planes of a reduced sphere and the circumcircle sphere is the tangential triangle of the base triangle (because the tangent to the circumcircle at a vertex of the triangle is tangent to both the reduced sphere and the circumcircle sphere therefore has the same power about both spheres).

The essential new property is the following

Theorem 3.1 *The centroid of the tetrahedron lies on the radical axis of the three reduced spheres.*

To prove this we use a Theorem given by Court in [1, § 187 p. 57] about the square of the median of a tetrahedron, if g_a is the length of the median going through A:

$$g_a^2 = \frac{1}{3}\left(A'I^2 + A'B^2 + A'C^2\right) - \frac{1}{9}\left(B'I^2 + C'I^2 + B'C'^2\right).$$

With our notation (fig. 1), we get

$$\begin{aligned} g_a^2 =& \frac{1}{3}\left((2x^2+r^2)+2(x^2+y^2)+2(x^2+z^2)\right) \\ & -\frac{1}{9}\left((2y^2+r^2)+(2z^2+r^2)+2(y^2+z^2)\right). \end{aligned}$$

Therefore, the square of the length of the median from A' is given by

$$g_a^2 = 2x^2 + \frac{2}{9}(y^2+z^2) + \frac{1}{9}r^2 \qquad (2)$$

and similarly.

We evaluate the power of the centroid of $IA'B'C'$, denoted G'', about the reduced sphere $\mathscr{S}'_{A'}$ centered at A':

$$\begin{aligned} \mathscr{P}\left(G''/\mathscr{S}'_A\right) =& \left(\frac{3}{4}g_a\right)^2 - x^2 \\ =& \frac{1}{16}\left(2(x^2+y^2+z^2)+r^2\right). \end{aligned}$$

If we compare this result with the square of the length of a bimedian (1), we can conclude that the sphere centered on the centroid and going through the midpoints of the side of the tetrahedron, i.e. the *first twelve point sphere* of this orthocentric tetrahedron [1, § 798 p. 300], is orthogonal to the three reduced spheres.

Therefore both the circumcircle sphere and the first twelve point sphere belong to the same intersecting coaxal pencil, conjugate of the coaxal net of spheres defined by the three reduced spheres. The plane of centers, A', B', C', of the coaxal net formed by the reduced spheres is the upper face of the tetrahedron. Now the trace on the upper plane of the first twelve point sphere being the Euler circle of the upper triangle this circle is the basic circle of the pencil. We have found that

Corollary 3.1 *The first twelve point sphere of the orthocentric tetrahedron of a triangle and the circumcircle sphere of this triangle have the same trace on the upper face of the tetrahedron: the Euler circle of the upper triangle,*

and, the line of centers of the pencil being parallel to the fourth altitude,

Corollary 3.2 *The Euler line of the orthocentric tetrahedron of a triangle belongs to the plane perpendicular to the upper face and going through the circumcenter and the incenter of the basic triangle.*

Lots of other properties are direct consequences of Theorem 3.1, e.g.

a) the orthogonal projection of the circumcenter of the base triangle on the upper plane is the center of the Euler circle, ω', of the upper triangle.

b) The orthogonal projection of the line OI of the base triangle ABC on the upper plane is the Euler line of the upper triangle $A'B'C'$. Both lines intersect on the Gergonne line of the base triangle.

Most of these properties are represented on Figure 9.

4 The 4^{th} Median and the Housel Line

We define the 4^{th} median of the tetrahedron as we defined its fourth altitude, the median going through the apex I. We now consider the medial triangle DEF (defined in [2, § 96 p. 68]) of the base triangle ABC, its incenter i, the Spieker Center [5, §364 p. 226], and the Housel line, IG, of ABC (defined in [3, p. 1260]) which is such that $\overrightarrow{IG} = 2\overrightarrow{Gi}$. We have the following property

Theorem 4.1 *The orthogonal projection of the centroid of the orthocentric tetrahedron of a triangle on the plane of this triangle lies on the*

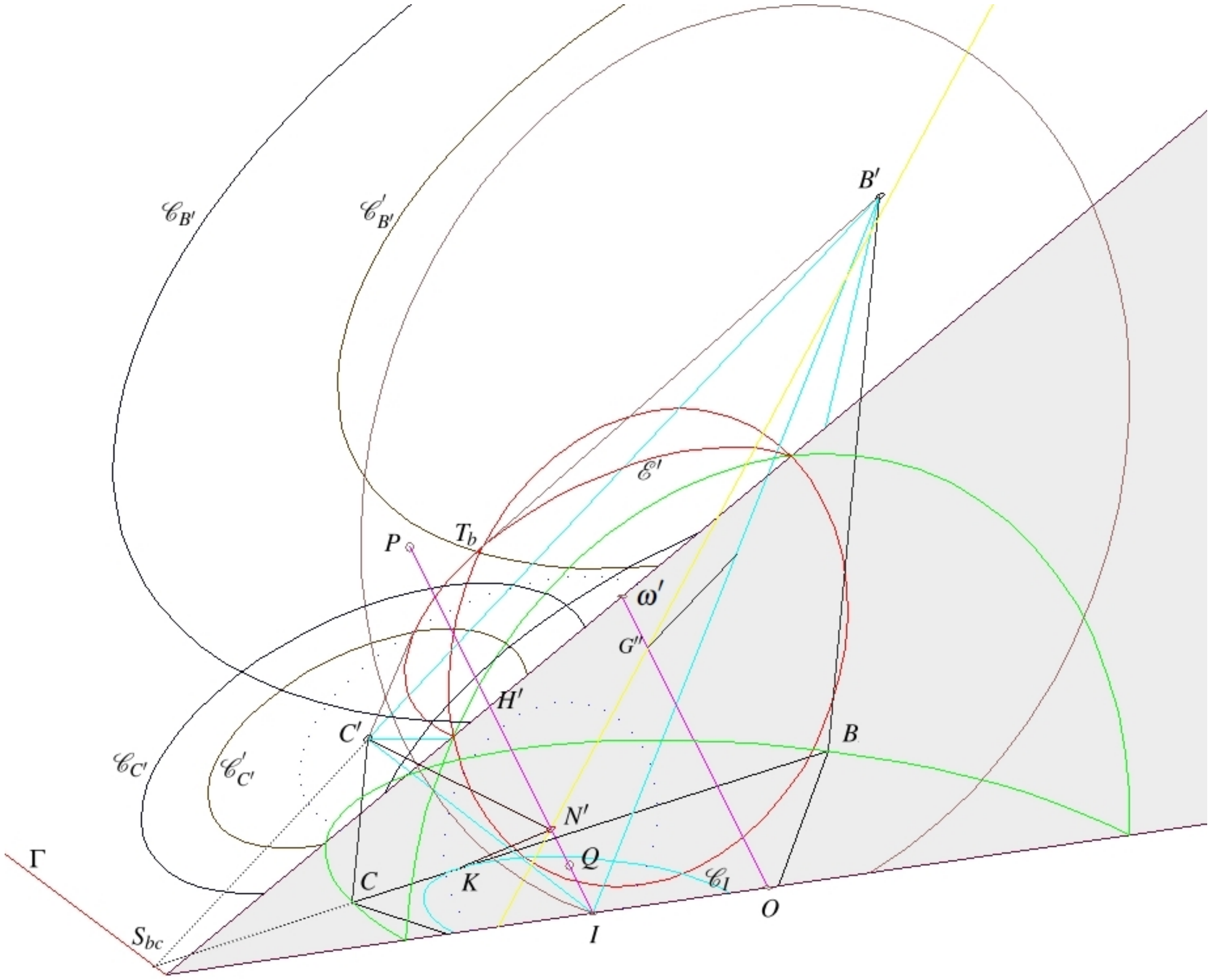

Figure 9: Cross-section along the plane OIH'

Housel line of the triangle and is the harmonic conjugate of the incenter of the medial triangle of this base triangle about its centroid and its incenter.

We call G' the centroid of the face $A'B'C'$ and g'' the projection of G'' on the base plane so that IG' is the fourth median of the tetrahedron (fig. 10). Because ABC is the orthogonal projection of $A'B'C'$, the orthogonal projection of G' on the base plane is G, the centroid of the base triangle. We have

$$\overrightarrow{IG''} = \frac{3}{4}\overrightarrow{IG'} \Rightarrow \overrightarrow{Ig''} = \frac{3}{4}\overrightarrow{IG},$$

therefore we have

$$\frac{g''G}{g''I} = \frac{1}{3} \text{ and } \frac{iG}{iI} = \frac{1}{3} \Rightarrow (I, G, g'', i) = -1.$$

We conclude that g'' is also the $X(1125)$ point in the Encyclopedia of Triangle Centers. It is also easy to check that i is the orthogonal projection on the plane of the base triangle ABC of the point i' reflexion of I in G'' and that the orthogonal projection of i' on the plane of the upper triangle is the circumcenter O' of $A'B'C'$. Of course, with the same notations, we could as well use the following proof and rediscover the property of the Housel line the following way

$$(H', G', \omega', O') = -1 \Rightarrow (I, G', G'', i') = -1$$

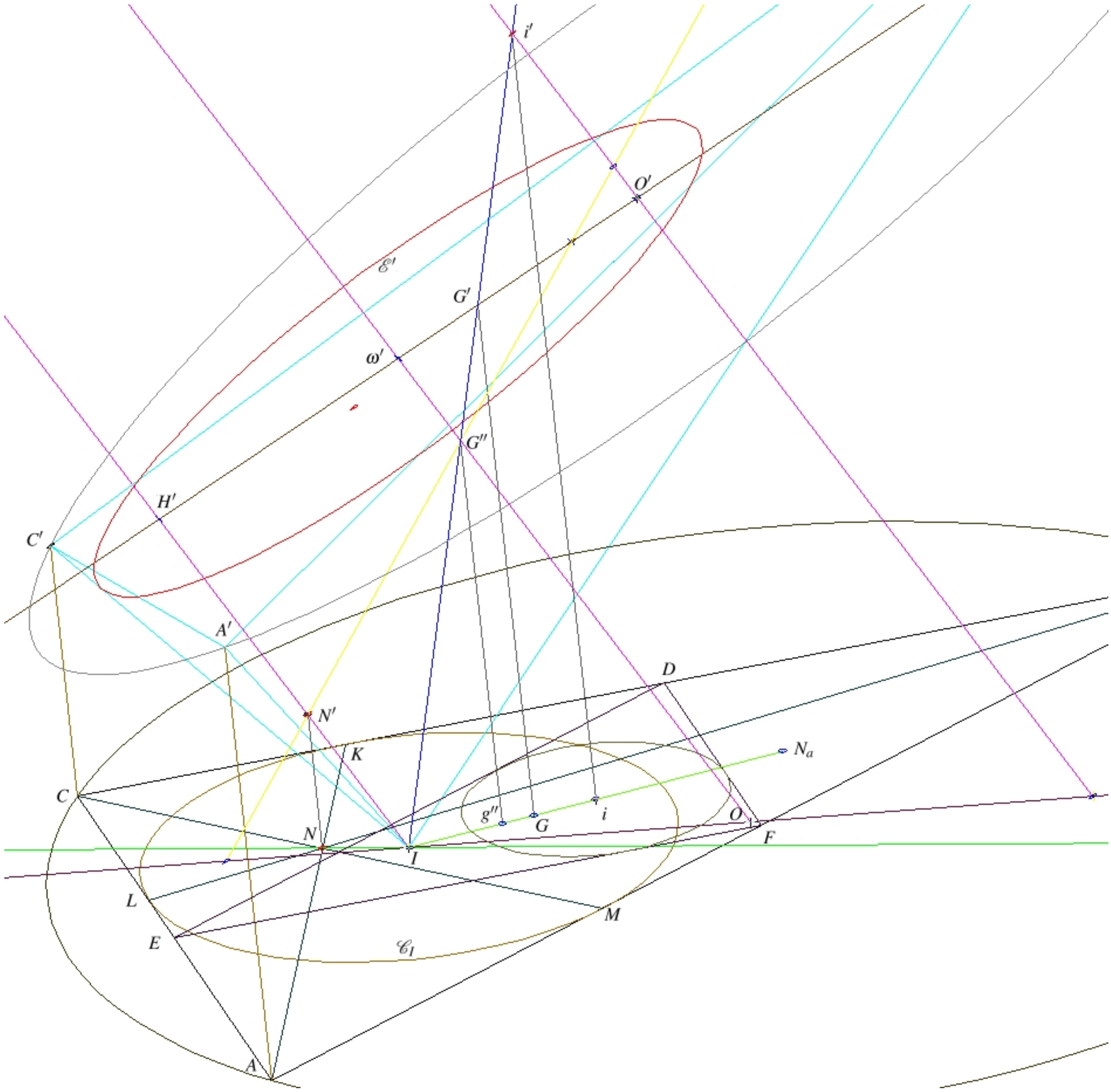

Figure 10: The 4^{th} Median and the Housel Line

$$\Rightarrow (I, G, g'', i) = -1 \Rightarrow \overrightarrow{IG} = 2\overrightarrow{Gi}.$$

Moreover on the line $O\omega'$, we will also find the centroid of the tetrahedron $N'A'B'C'$ and the first twelve point sphere of this tetrahedron belongs to the coaxal pencil defined by the circumcircle sphere and first twelve point sphere of $IA'B'C'$.

5 Triangle geometry

We find in [5, §310 p. 197] the following theorem (applied to the triangle $A'B'C'$),

Theorem 5.1 *The sum of the powers of the vertices with regards to the Euler circle is* $\frac{1}{4}\sum a'^2$.

From fig. 1 we have, applying Pythagoras, twice, $a'^2 = 2(y^2 + z^2)$ so that

$$\sum a'^2 = \sum 2(y^2 + z^2) = 4\sum x^2. \qquad (3)$$

Now, because the Euler circle, $\mathscr{E}'$, of the upper triangle is orthogonal to the three reduced circles $\mathscr{C}'_{A'}$, radius x, ..., we find that

$$\mathscr{P}\left(A'/\mathscr{E}'\right) = x^2 = \mathscr{P}\left(A/\mathscr{C}_I\right).$$

Hence we have the

Theorem 5.2 *The power of a vertex of the upper triangle with regard to its Euler circle is equal to the power of the corresponding vertex of the base triangle with regard to the incircle.*

Therefore, from (3),

$$\sum \mathscr{P}\left(A'/\mathscr{E}'\right) = \sum x^2 = \frac{1}{4}\sum a'^2.$$

This is a new proof of Theorem 5.1 but it also gives us a very different insight about this theorem.

Remark 5.1 *This gives an interesting solution to the following question of plane triangle geometry:*

Given a triangle, its Euler circle and a set of three circles centered on the three vertices orthogonal to the Euler circle, show that if we multiply the radii of the three circles by $\sqrt{2}$*, then the new set of circles has its radical center at the orthocenter of the triangle.*

6 Inverse problem

To construct the triangle ABC, we draw the Euler circle, center ω', of the upper triangle, three tangents to this circle from the vertices A', B', C' (e.g. $B'T_b$ fig.11). It gives the radii of the reduced circles (e.g. $y = B'T_b$) and therefore the three reduced circles and spheres (e.g. $\mathscr{S}'_{B'}$).

Remark 6.1 *Note that if the triangle is obtuse one of the vertex is inside the Euler circle and as announced, the problem has no real solution. If it is right we fall back on the degenerate case we started with in the extended abstract.*

We construct two of the external centers of similitudes S_{bc} and S_{ca} of $\mathscr{C}'_{A'}, \mathscr{C}'_{B'}$ and $\mathscr{C}'_{C'}$, it gives Γ, the Gergonne line of ABC (§3). The plane tangent to any of these three spheres and going through this Gergonne line is the plane of the base triangle ABC. The points of tangency of the base plane with each of the reduced spheres are the three vertices A, B, C we are looking for.

To get them, we use the *section* (of a solid by a plane) command of a CAD software and we consider the pencil of planes having the Gergonne line for axis.

- The trace of this pencil on one of the sphere is a coaxal pencil of circles with limiting points on the sphere [8, §967 p. 280].
- Therefore using a line Δ_p perpendicular to the plane $A'B'C'$ (e.g. at B' fig. 12) through the center of the sphere, we take a point $p1$ on this line, inside the sphere, and we sec-

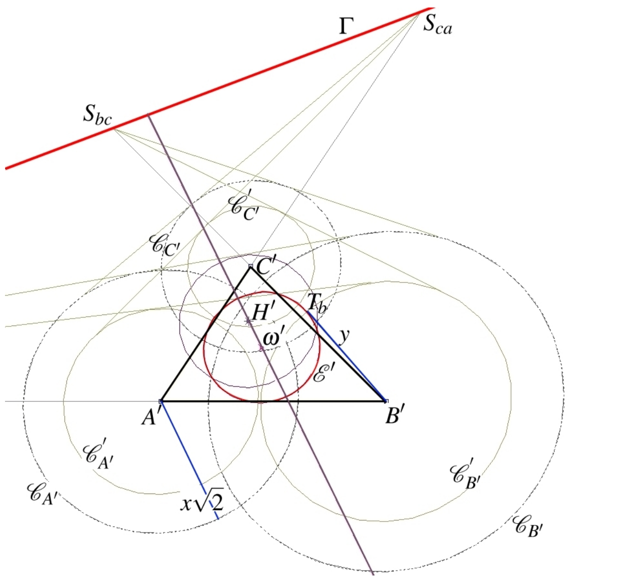

Figure 11: Three reduced circles and the D'Alembert line

tion the sphere by the plane $p1$, S_{bc} and S_{ca} to get a circle $\mathscr{C}_{p1}$.

- We mark the center C_{p1} of this circle. We do this with a second point $p2$, marking the center C_{p2}.
- We section again the sphere with the plane going through B', the center of the sphere, and the two marked centers C_{p1}, C_{p2}. We get a new circle $\mathscr{C}_{cb'}$.
- The intersection B_u (or B_d) of this circle with the circumcircle of the triangle B', C_{p1}, C_{p2} gives the point of tangency as an orthogonal intersection (fig. 13).

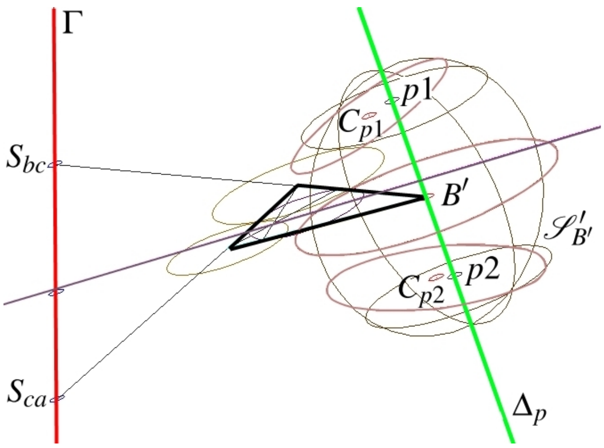

Figure 12: Pencil of circles on the sphere $\mathscr{S}'_{B'}$

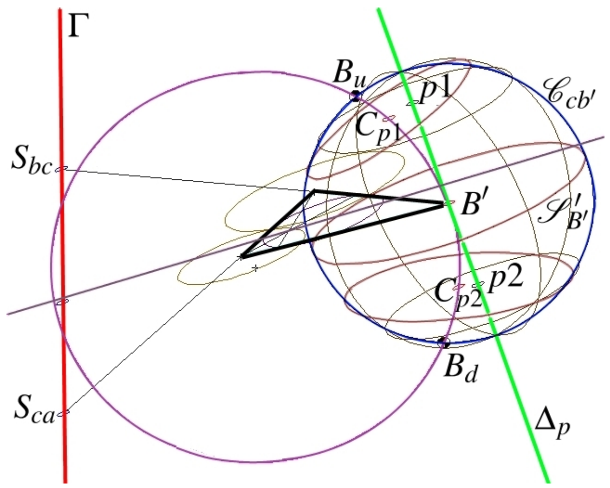

Figure 13: Construction of the 2 symmetrical points of tangency

Doing this operation for the three vertices gives us A, B, C and from there we get I, a very easy solution to the inverse problem. Up to a symmetry about the plane of the upper face, the solution is unique.

7 Conclusions

Associating tri-dimensional objects to a triangle is a fruitful operation, we could call this "'*Solid Triangle Geometry*"'. In [7] we associated an orthogonal tetrahedron and the incircle sphere centering the research on the fourth altitude, here adding the circumcircle sphere, we have proved new properties linked to the Euler lines and the 4^{th} median. They show how connected the triangle is with its orthocentric tetrahedron. Using some of these new properties we derive a construction procedure for the inverse problem and show that, up to a symmetry, we have a single solution.

The correspondence between any base triangle and the acute upper triangle of its orthocentric tetrahedron being one to one, we have found that an arbitrary orthocentric tetrahedron is not necessarily the orthocentric tetrahedron of a triangle. For a given acute triangle, there is only two symmetrical points lying on the perpendicular to its plane going through its orthocenter which gives the apex of such a tetrahedron. If it does, the property would, in the general case, apply to a single face of this tetrahedron.

Finally, though Euclidean Solid Geometry is a very mature discipline, this paper illustrates that there are potentially a lot of beautiful properties left to discover and the use of software may facilitate this process.

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About the author

Axel Pavillet is a graduate of Ecole Polytechnique in Paris and has a M. Eng in industrial engineering from Ecole Nationale Supérieure des Techniques Avancées. He spent most of his career as a professional engineer for the French Government and worked in France, the US and Argentina. He immigrated to Canada in 1998 and received a M.Sc. in Computer Science and a Ph.D. in Mathematics from UQAM (Université du Québec à Montréal) in 2001 and 2004.

He has published numerous articles in the scientific and technological fields. He is knight of the National Order of Merit from France and was awarded the Meritorious Service Medal from the United States.