The orthocentric tetrahedron of a triangle

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Solid and Triangle Geometry

Abstract

A triangle is a geometric structure which has a wealth of geometric objects attached to it, thousands of points and hundreds of two dimensional objects such as lines, circles, other conics and then some. In this article we attach to a triangle, probably for the first time, a three dimensional object: an orthocentric tetrahedron. We give some of its properties and show the deep connections between this tetrahedron and classical theorems of triangle geometry.

1 A triangle and its associated tetrahedron

We consider a triangle ABC on the horizontal plane, called the base triangle, and its inscribed circle $\mathscr{C}_I$. The point of contact of the incircle with the sides are K, L, M and I the incentre.

From A, B and C we draw the vertical segments $AA' = AM$, $BB' = BK$, $CC' = CL$ and we consider the tetrahedron $A'B'C'I$ (fig. 1, using notations from [6, § 1.4, p. 11]). From the figure we have

$$A'A \perp AM,\ AM \perp IM \Rightarrow A'I^2 = A'A^2 + AM^2 + MI^2 = 2x^2 + r^2$$

and also

$$\angle AMA' = \angle B'MB = \frac{\pi}{4} \Rightarrow A'M \perp B'M \Rightarrow A'B'^2 = A'A^2 + AM^2 + MB^2 + BB'^2 = 2x^2 + 2y^2$$

and similarly for the other sides. Therefore we have

$$A'I^2 + B'C'^2 = 2(x^2 + y^2 + z^2) + r^2.$$

it implies that the three sums of the squares of the three pairs of opposite edges of this tetrahedron are equal. Now this is a property which, usually, is linked to an orthocentric tetrahedron [2, §211 p. 63], we prove

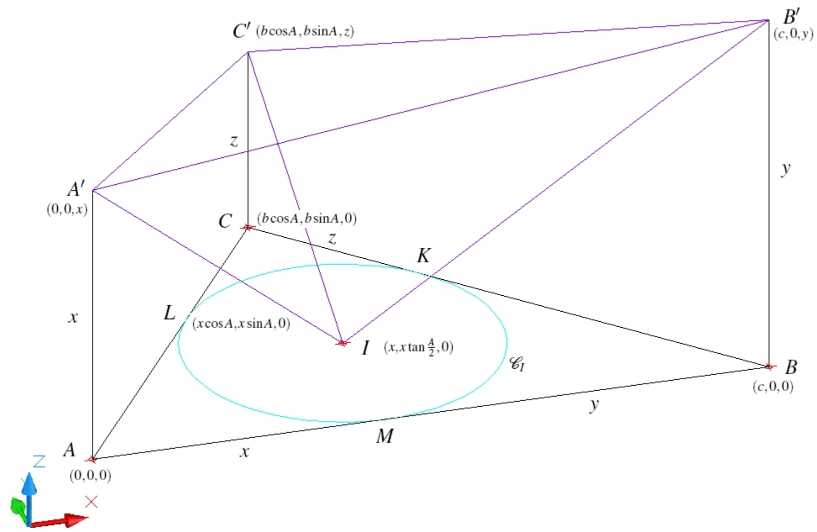


Figure 1: notations

Theorem 1.1 *The tetrahedron $A'B'C'I$ is orthocentric.*

Proof:

With the notation of the figure, we have

$$\left\{\begin{array}{ccccl} x & + & y & = & c; \quad x-y=b-a \\ y & + & z & = & a; \quad y-z=c-b \\ z & + & x & = & b; \quad z-x=a-c \end{array}\right. \tag{1}$$

and the coordinates of the various points we need are

A	$(0,0,0)$	A'	$(0,0,x)$	I	$(x, x\tan\frac{A}{2}, 0)$
B	$(c,0,0)$	B'	$(c,0,y)$	L	$(x\cos A, x\sin A, 0)$
C	$(b\cos A, b\sin A, 0)$	C'	$(b\cos A, b\sin A, z)$		

We have

$$\overrightarrow{A'I} = x\vec{i} + x\tan\frac{A}{2}\vec{j} - x\vec{k} \Rightarrow \frac{1}{x}\overrightarrow{A'I} = \vec{i} + \tan\frac{A}{2}\vec{j} - \vec{k}$$

and

$$\overrightarrow{B'C'} = (b\cos A - c)\vec{i} + b\sin A\vec{j} + (z-y)\vec{k}$$

therefore

$$\frac{1}{x}\overrightarrow{A'I}\cdot\overrightarrow{B'C'} = b\cos A - c + b\sin A\tan\frac{A}{2} + y - z = b\left(2\cos^2\frac{A}{2} - 1\right) - c + b\cdot 2\sin\frac{A}{2}\cos\frac{A}{2}\tan\frac{A}{2} + y - z$$

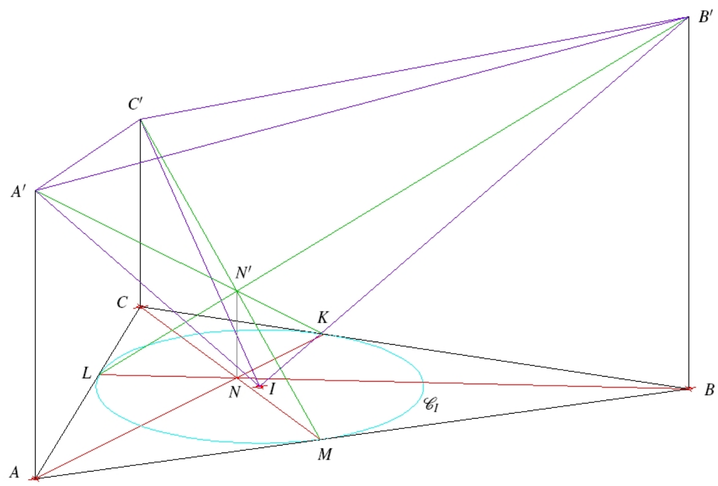


Figure 2: three altitudes

so that from (1)

$$b-c+y-z=0.$$

From this computation we conclude that $A'I \perp B'C'$ but also that, similarly, $B'I \perp C'A'$ and $C'I \perp A'B'$. The three pair of edges are orthogonal which is enough to validate the theorem.

QED.

Corollary 1.1 *The length m of the bimedian of the tetrahedron is given by*

$$4m^2 = A'I^2 + B'C'^2 = 2(x^2+y^2+z^2)+r^2. \tag{2}$$

Remark 1.1 *In [2] N. A. Court does not prove the converse of the theorem concerning the three sum of the squares of opposite edges. This proof is not difficult but its computation is longer than this direct proof.*

Theorem 1.2 *The segments $A'K$, $B'L$ and $C'M$ are three altitudes of the tetrahedron $A'B'C'I$.*

Proof:

We have (fig. 2)

$$\overrightarrow{B'L} = (x\cos A - c)\vec{i} + x\sin A\vec{j} - y\vec{k}$$

which is

$$\frac{1}{x}\overrightarrow{A'I}\cdot\overrightarrow{B'L} = (x\cos A - c) + x\sin A\tan\frac{A}{2} + y = x\left(2\cos^2\frac{A}{2}-1\right) - c + 2x\sin^2\frac{A}{2} + y$$

so that

$$x - c + y = 0.$$

Therefore $B'L \perp A'I$. Similarly, we will get $B'L \perp C'I$ so that $B'L$ is perpendicular to the plane $A'C'I$ and is an altitude of the tetrahedron.

With the same computation we will get $C'M \perp B'I$ and so on ....

QED.

Now the tetrahedron being orthocentric, the three altitudes intersect at a point N' which is the orthocentre of the tetrahedron. The plane $AA'K$ and its siblings are vertical and intersect along a vertical line, the trace of this line on the horizontal plane is at the intersection of the Cevians AK, BL, CM (defined in [3, § 329, p. 160]). This point N is the Gergonne point of the triangle (from [3, § 331, p. 160]).

We have proved

Theorem 1.3 *The orthogonal projection of the orthocentre of the orthocentric tetrahedron of a triangle on the plane of its triangle is the Gergonne point of this triangle.*

Note that this also proves that these Cevians are concurrent. It is usually done using Ceva's theorem [6, p. 13].

Now the fourth altitude is the line IN', therefore, if, and only if, the triangle is not equilateral its projection on the horizontal plane is the line IN: this is the so-called *Soddy line*.

Remark 1.2 *If the triangle is equilateral, its Soddy line is not defined because I and N are coincident, and here the fourth altitude is vertical, its projection on the plane ABC is the point I. This case is not totally trivial because one can check that the orthocentric tetrahedron of an equilateral triangle is simultaneously orthocentric, isodynamic, isogonic, and circumscriptible.*

- orthocentric*, obviously*
- isodynamic*, the three products of the three pairs of opposite edges are equal [2, §835]*
- isogonic*, by symmetry, the lines joining the vertices of this tetrahedron to the point of contact of the opposite face with the inscribed sphere are concurrent on the fourth altitude [2, §879]*
- circumscriptible*, the three sums of the three pairs of opposite edges are equal [2, §790]*

2 The tritangent sphere

We need some constructions before going further.

From the triangle ABC, we draw the circles $\mathscr{C}_A$, centre A, radius $x = AM$ and its two siblings. We call V, the centre of the inner Soddy circle. This circle $\mathscr{C}_V$, is tangent to the three circles $\mathscr{C}_A$, $\mathscr{C}_B$ and $\mathscr{C}_C$.

We consider the composite surface formed by three auxiliary right half-cones with vertex A', B', C' and base $\mathscr{C}_A, \mathscr{C}_B, \mathscr{C}_C$.

Finally, to this set of solids, we add a sphere $\mathscr{S}_v$ which is tangent to the three cones along the inner Soddy circle. We call V' the centre of this tritangent sphere. If U is the point of contact of the inner Soddy circle with $\mathscr{C}_A$ (fig. 3), a cross-section along the generatrix $A'U$ shows that V, V' and the point V'' on $A'U$ are collinear on a vertical line with $VV' = VV''$.

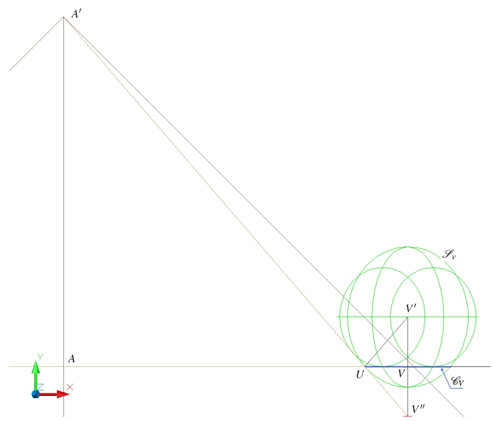


Figure 3: The tritangent sphere

3 The fourth altitude

With this, we can study the properties of the line IN', the fourth altitude of the tetrahedron.

We consider the tritangent sphere $\mathscr{S}_v$ and three vertex spheres, $\mathscr{S}_A$ centered on A' with radius $A'M = x\sqrt{2}$ and similarly $\mathscr{S}_B$ centered on B' with radius $B'K$, $\mathscr{S}_C$ centered on C' with radius $C'L$.

- The set $\{\mathscr{S}_A, \mathscr{S}_B, \mathscr{S}_C\}$ is a triad of mutually orthogonal spheres because

$$A'M \perp B'M \Rightarrow \mathscr{S}_A \perp \mathscr{S}_B.$$

- The spheres $\mathscr{S}_A$ and $\mathscr{S}_v$ are orthogonal. This is because $\mathscr{S}_v$ being tangent to the cone based on $\mathscr{C}_A$ at U we have $A'U \perp V'U$.

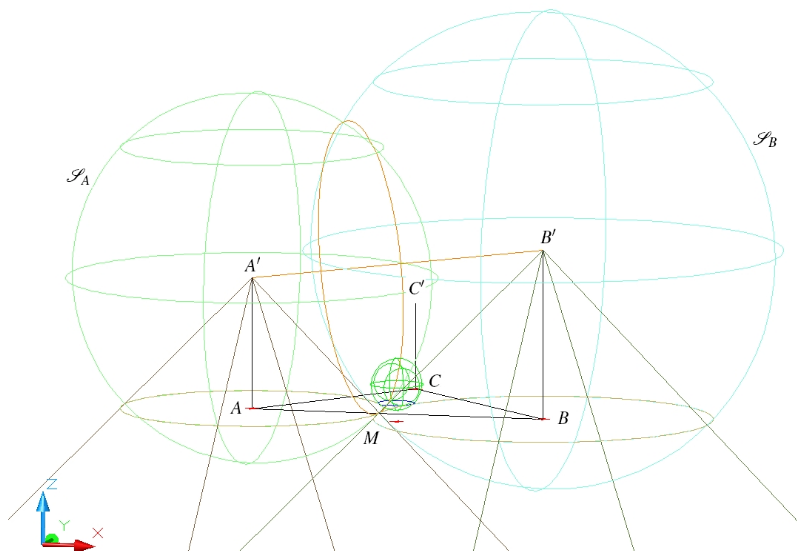
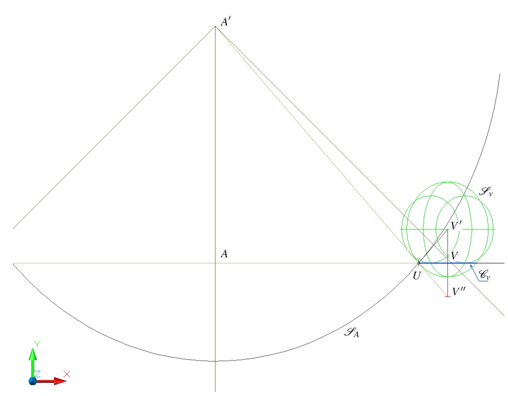


Figure 4: the radical plane of two of the vertex spheres



- Therefore $\mathscr{S}_A$, $\mathscr{S}_B$ and $\mathscr{S}_C$ being orthogonal to $\mathscr{S}_v$ they form an *intersecting* coaxal net of spheres [2, p. 220 § 602-604 (f)]
- and the set $\{\mathscr{S}_v, \mathscr{S}_A, \mathscr{S}_B, \mathscr{S}_C\}$ is a tetrad of mutually orthogonal spheres.

Let's consider two of the vertex spheres, e.g. $\mathscr{S}_A$ and $\mathscr{S}_B$.

- We call Π_c the radical plane of $\mathscr{S}_A$ and $\mathscr{S}_B$, it is perpendicular to the segment $A'B'$.
- The opposite vertex C' belongs to this radical plane because $\mathscr{S}_C$ being orthogonal to $\mathscr{S}_A$ and $\mathscr{S}_B$ the power of C' about each of them is the square of the radius of $\mathscr{S}_C$, $2z^2$ so $\mathscr{P}(C'/\mathscr{S}_A) = \mathscr{P}(C'/\mathscr{S}_B) = 2z^2$.

- The point M belongs to $\mathscr{S}_A$ and $\mathscr{S}_B$. We conclude that the altitude $C'M$ of the tetrahedron belongs to the radical plane of $\mathscr{S}_A$ and $\mathscr{S}_B$.

Therefore, this property being valid for the three spheres, the orthocentre of the tetrahedron belongs to the radical axis of the net.

Now the incentre of ABC too belongs to this axis because the power of I about any of the three spheres is $IM^2 = IK^2 = IL^2 = r^2$, the square of the radius of the incircle.

Finally V', the centre of $\mathscr{S}_v$ is also on the axis because $\mathscr{S}_v$ being orthogonal to the three spheres, the power of V' about any of the spheres of the net is the square of the radius of $\mathscr{S}_v$.

We have proved

Theorem 3.1 *The centre of the incircle, the centre of the tritangent sphere and the orthocentre of the tetrahedron are collinear.*

Projecting on the horizontal plane, we also have given a new proof of the main property of the Soddy line:

Corollary 3.1 *The centre of the incircle, the centre of the Soddy circle and the Gergonne point of a triangle are collinear.*

On the fourth altitude, we also find P and Q, the base points of the coaxal net defined by the spheres $\mathscr{S}_A, \mathscr{S}_B, \mathscr{S}_C$ and H' the orthocentre of the triangle $A'B'C'$, the centre of the net and middle point of the segment PQ.

Remark 3.1 *Note that we built the tetrad of mutually orthogonal spheres $\{\mathscr{S}_v, \mathscr{S}_A, \mathscr{S}_B, \mathscr{S}_C\}$ from a set of circles $\{\mathscr{C}_v, \mathscr{C}_A, \mathscr{C}_B, \mathscr{C}_C\}$ mutually externally tangent. Therefore, because all the radii of the spheres are $\sqrt{2}$ times the radius of the corresponding circles, the four radii of the spheres are linked by the same relation as the one linking the radii of the circles given by Descartes Theorem [5, p. 14]. If σ are the algebraic values of the radii of the Soddy circles (> 0 for the inner Soddy circle, possibly < 0 for the outer one), we have*

$$2\left(\frac{1}{\sigma^2}+\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right)=\left(\frac{1}{\sigma}+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^2. \tag{3}$$

and if $R_a, \ldots R_v$ are the radii of the mutually orthogonal spheres, we have a similar formula:

$$2\left(\frac{1}{R_v^2}+\frac{1}{R_a^2}+\frac{1}{R_b^2}+\frac{1}{R_c^2}\right)=\left(\frac{1}{R_v}+\frac{1}{R_a}+\frac{1}{R_b}+\frac{1}{R_c}\right)^2.$$

Remark 3.2 *The same tetrahedron can be drawn by symmetry on the other side of the plane ABC, in that case the fourth altitude goes through the point V'' and the vertex spheres are inscribed in the cones along the base circles.*

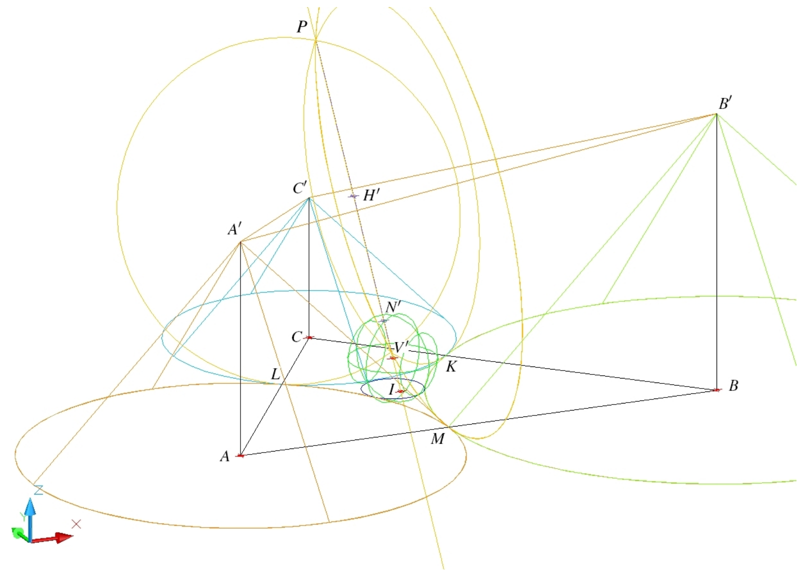


Figure 5: The fourth altitude goes through V'

We will call the vertical plane INN', the principal plane of the orthocentric tetrahedron $A'B'C'I$, the plane of the triangle $A'B'C'$ the upper plane of the tetrahedron and $A'B'C'$ its upper triangle. The fourth altitude IN' of the tetrahedron is perpendicular to the upper face. Therefore, the Soddy line IN, its projection on the horizontal plane, is perpendicular to the trace of the upper face on the horizontal plane (fig. 6) [12, T II - p. 20 §533]. Let Δ be this line with $S_{ab} = AB \cap \Delta$ and similarly. Because the orthogonal projection of $A'B'C'$ on the horizontal plane is ABC the S_{ab} points are also the intersection of the corresponding $A'B'$ lines with Δ.

Now $\frac{S_{ca}A}{S_{ca}C} = \frac{S_{ca}A'}{S_{ca}C'} = \frac{AA'}{CC'} = \frac{x}{z}$ so that the points S_{ab} are the external centres of similitude of the three circles $\mathscr{C}_A, \mathscr{C}_B$, and $\mathscr{C}_C$ taken in pairs.

It implies that the external centre of similitude S_{ca} is also the harmonic conjugate of L about AC, and similarly for M and N so that Δ is the Gergonne line of the triangle, it proves that

Theorem 3.2 *The traces of the sides of the upper triangle of the orthocentric tetrahedron on the plane of the base triangle are the harmonic conjugates relative to the vertices of base triangle of the point of contact of the incircle.*

and

Theorem 3.3 *The trace of the upper face of the orthocentric tetrahedron of a triangle on the plane of the base triangle is its Gergonne line.*

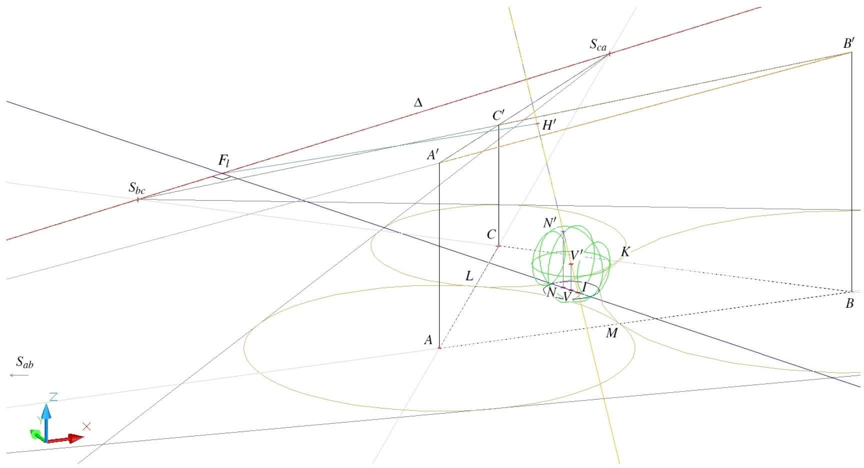


Figure 6: IN is perpendicular to Δ

It also gives another proof of the following property [10, p. 324 §6]:

Corollary 3.2 *The Soddy line is perpendicular to the Gergonne line.*

The intersection of the Gergonne line with the principal plane of the tetrahedron, F_l on figure 6 could be called the *radical origin* of ABC, since it's the intersection of the radical axes and lines of centres of two noteworthy coaxal systems of ABC[1].

Finally, the points A', C', K, M and S_{ca} are coplanar because the lines $A'C'$ and KM intersect at S_{ca}. Therefore the lines $A'M$ and $C'K$ intersect. In the vertical plane $AA'BB'$, let B'' be the point $A'M \cap BB'$. Because the slopes of $A'M$ and $B'M$ in this plane are $\left|\frac{\pi}{4}\right|$ we have $BB'' = BB'$. The same property will apply for $C'K$ hence $A'M$ and $C'K$ intersect at B'' symmetrical of B' about the plane of the base triangle. We can similarly find the points A'' and B''. It gives the tetrahedron $A''B''C''I$. Now this is also a consequence of the fact that the tetrahedron could be drawn on the other side of the base triangle (cf. remark 3.2).

4 Perspective triangles

We know that the triangles ABC and its contact triangle KLM are in perspective. The axis of perspective being the Gergonne line and the centre of perspective being the Gergonne point of the triangle. Now, because ABC is the orthogonal projection of $A'B'C'$, these two triangles also are in perspective. The centre of perspective is the point at ∞ on the axis Oz and we have also seen that the axis of perspective was the Gergonne line

[1]We follow here a suggestion of Richard Guy.

of the triangle ABC. It is also clear that $A'B'C'$ and KLM are in perspective, the centre of perspective being the orthocentre of the tetrahedron while we keep the Gergonne line as the single axis of perspective. This generalizes the property given in [10, §6 p. 327].

5 The incircle sphere

We define the incircle sphere, $\mathscr{S}_I$, as the sphere which has the incircle of the base triangle as its great circle.

The tetrahedron $A'B'C'I$ being orthocentric, the five points $A'B'C'IN'$ form an orthocentric group [2, p. 310, § 826] and I is the orthocentre of $N'A'B'C'$. We have seen that the power of I relative to the three spheres was r^2, but r is the radius of the incircle sphere therefore the incircle sphere is orthogonal to these three spheres and like the tritangent sphere, $\mathscr{S}_I$ belongs to the coaxal pencil of spheres with limiting points P and Q conjugate of the coaxal net formed by the three spheres $\mathscr{S}_A, \mathscr{S}_B, \mathscr{S}_C$.

Now the radical plane of this coaxal pencil is the upper face of the tetrahedron. Limiting our view to this coaxal pencil of spheres and its radical plane, we can intersect it with the plane of the base triangle; we get a coaxal pencil of circles with limiting point. The traces of the pencil on this plane are two circles, the incircle for the incircle sphere and the inner Soddy circle for the tritangemt sphere. Because the intersection of the radical plane with the base plane is the Gergonne line, we have a new proof of

Theorem 5.1 *The Gergonne line is the radical axis of the coaxal pencil of circles formed by the incircle and the inner Soddy circle.*

Let's again consider Π_c the radical plane of $\mathscr{S}_A$ and $\mathscr{S}_B$. It goes through C' and is perpendicular to $A'B'$ (fig. 7), it also contains $C'M$, the altitude of the tetrahedron going through C' and the altitude of the upper triangle going through the same vertex. Therefore it contains H' the orthocentre of the upper triangle and N' the orthocentre of the tetrahedron. The traces of the spheres $\mathscr{S}_I$ and $\mathscr{S}_B$ on Π_c are two circles which have in common M and another point M_1. These two circles have C' on their radical axis because the power of C' relative to both spheres is $2z^2$ as seen above. Therefore $C'M$ is the radical axis of the two circles of intersection.

The spheres $\mathscr{S}_A, \mathscr{S}_B$, and $\mathscr{S}_C$ are orthogonal, therefore their great circles $\mathscr{C}'_A, \mathscr{C}'_B$, and $\mathscr{C}'_C$ are orthogonal in the plane $A'B'C'$ (fig. 8) and the radical centre of these three great circles is the orthocentre H'. It implies that the polar of C' about the great circle of $\mathscr{S}_B$ is the altitude $A'H'$ going through A' and therefore the polar plane of C' about $\mathscr{S}_B$ is the plane through $A'H'$ perpendicular to $C'B'$: it is the plane $A'H'N'$.

We conclude that C' and N' are harmonic conjugate about the points of intersection M and M_1 of $C'M$ with $\mathscr{S}_I$ and therefore the polar plane of C' about $\mathscr{S}_I$ is going through N' and is perpendicular to $C'I$. But $IN'A'B'C'$ form an orthocentric group [2,

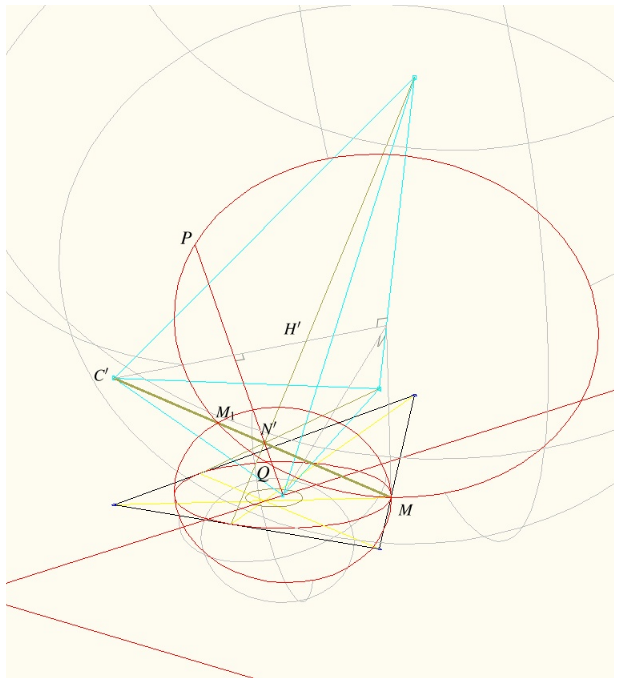


Figure 7: $C'M$ is the radical axis of the two circles of intersection

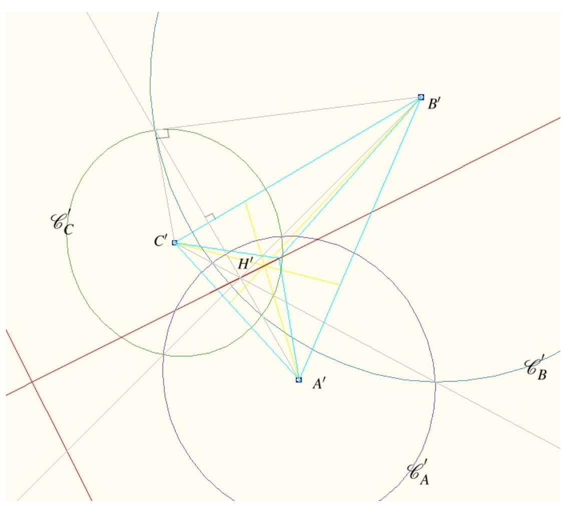


Figure 8: $A'H'$ is the polar of C'

§826] consequently the plane $N'A'B'$ is this polar plane. This is valid for the other two vertices and then the polar plane of N' has to be $A'B'C'$.

Using the definition of [2, §795], we have proved

Theorem 5.2 *The incircle sphere is the polar sphere of the orthocentric tetrahedron $N'A'B'C'$.*

A direct consequence of this theorem is that $(I,N',P,Q) = -1$, N' being the radical center of the four spheres $\mathscr{S}_A, \mathscr{S}_B, \mathscr{S}_C$, and $\mathscr{S}_I$ ([2, § 627]).

Now, we have shown that the polar plane of C' about $\mathscr{S}_B$ is the plane $A'H'N'$ or equivalently $A'IN'$, it implies that $\mathscr{S}_A, \mathscr{S}_B$, and $\mathscr{S}_C$ are three other polar spheres from the orthocentric group. The fifth sphere from this group is the imaginary sphere $\mathscr{S}_{N'}$ centered on N' and therefore because $\mathscr{S}_A, \mathscr{S}_B, \mathscr{S}_C$, and $\mathscr{S}_I$ are always real, we have proved [2, § 796] that

Theorem 5.3 *The orthocentre of the orthocentric tetrahedron of a triangle is always inside the tetrahedron.*

Using Theorem 5.2 again, the polar plane of N' about the incircle sphere is the upper face $A'B'C'$. Now the polar plane of the point at ∞ in the direction Oz about the same sphere is the horizontal plane ABC. Therefore the intersection of these two planes is the line conjugate of the line NN' about the incircle sphere (defined as in [2, § 440 p. 144]).

We have proved

Theorem 5.4 *The line which is perpendicular to the plane of the base triangle and going through the orthocentre of the orthocentric tetrahedron of this triangle is the conjugate of its Gergonne line about the incircle sphere.*

Of course this is also a new proof of

Theorem 5.5 *The Gergonne point and the radical origin of a triangle are inverse points with respect to the incircle.*

6 The principal plane and the invariant relation

The principal plane is the vertical plane containing the Soddy line and the fourth altitude. We will call θ the angle between these two lines (fig. 10) so that the angle of the upper face with the base of the base triangle is $\frac{\pi}{2} - \theta$.

We use the Soddy line, origin at I, orientation positive in the direction of the Gergonne point or the centre of the inner Soddy circle and similarly for the fourth altitude.

The coordinates of all points and length of segments in this plane have an expression symmetrical in $x, y.z$.

Therefore we will use the following symmetrical notations:

- semiperimeter $s = x + y + z = \frac{1}{2}(a+b+c)$
- product pair $p = \sum xy = xy + yz + zx$
- product triple $t = xyz$
- radius of the incircle $r^2 = \frac{t}{s}$
- area of the base triangle $\Delta = s \cdot r$ and $\Delta^2 = t \cdot s$.
- $\sum x^2 = s^2 - 2p$
- $\sum x^2 \cdot y^2 = p^2 - 2ts$.

Some more notations:

- Length of the fourth altitude $IH' = h$.
- Segment $IN = g$ (incircle centre - Gergonne point), $g > 0$ due to the chosen orientation.
- Altitude of N', or length of the segment $NN' = n$, (orthocentre - Gergonne point).
- Segment $IV = v$ (incircle centre - centre of the inner Soddy circle), $v > 0$ due to the chosen orientation..
- Segment $IW = w$ (incircle centre - centre of the outer Soddy circle), can be positive or negative.
- Segment $IF_l = f$ (incircle centre - radical origin) and from theorem 5.5, N and F_l inverse points, we have $f \cdot g = r^2$.

We first compute $NN' = n$ as the $z-$coordinate of the point of intersection of the line $C'M$ and $B'L$. The notations are from Theorem 1.1.

The parametric equations of $B'L$ and $C'M$ are

$$C'M \left\{ \begin{array}{lclcl} X & = & x & + & \lambda((x+z)\cos A - x) \\ Y & = & & & \lambda(x+z)\sin A \\ Z & = & & & \lambda z. \end{array} \right. \qquad B'L \left\{ \begin{array}{lclcl} X & = & x+y & + & \mu(x\cos A - x - y) \\ Y & = & & & \mu x \sin A \\ Z & = & y & - & \mu y. \end{array} \right.$$

These two lines intersect at N', at this point $Z = n$ and we have, from $C'M$, $\lambda = \frac{n}{z}$ therefore $Y = \frac{n}{z}(x+z)\sin A$. From $B'L$ we get $\mu = \frac{y-n}{y}$ and $Y = \frac{y-n}{y}\sin A$. So that the Y coordinate of N' satisfies

$$\frac{n}{z}(x+z)\sin A = \frac{y-n}{y}\sin A \Rightarrow \frac{n}{z}(x+z) = \frac{y-n}{y}.$$

Solving for n yields

$$\frac{1}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \Rightarrow n = \frac{t}{p}. \tag{4}$$

We now compute $IN = g$ using some of the properties of our set of five orthogonal spheres: $\{\mathscr{S}_A, \mathscr{S}_B, \mathscr{S}_C, \mathscr{S}_I\}$ and the imaginary sphere $\mathscr{S}_{N'}$. Let ρ be its radius, it satisfies the following relation [4, Theorem 5 p.256]

$$\sum_{i=1}^{5} \frac{1}{r_i^2} = 0 \Rightarrow \frac{1}{\rho^2} + \frac{1}{2x^2} + \frac{1}{2y^2} + \frac{1}{2z^2} + \frac{1}{r^2} = 0$$

therefore

$$-\frac{1}{\rho^2} = \frac{1}{2x^2} + \frac{1}{2y^2} + \frac{1}{2z^2} + \frac{1}{r^2} = \frac{1}{2}\frac{\sum x^2 \cdot y^2}{x^2 y^2 z^2} + \frac{1}{r^2}.$$

Using our symmetric notations yield

$$-\frac{1}{\rho^2} = \frac{1}{2}\frac{p^2 - 2ts}{t^2} + \frac{s}{t} = \frac{1}{2}\frac{p^2}{t^2} \Rightarrow \rho = i\sqrt{2}n.$$

Now, this also means that the real sphere centered on N' with radius $-i\rho$ will be bisected by the incircle sphere [2, § 533-535]. Hence, a cross-section along the principal plane, (figure 9) yields

$$Iq^2 = r^2 = IN'^2 + N'q^2 = IN^2 + NN'^2 - \rho^2 \Rightarrow g^2 + n^2 - \rho^2 = r^2.$$

We have found an invariant relation of the orthocentric tetrahedron of a triangle

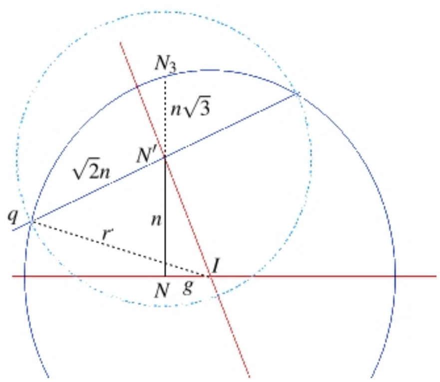


Figure 9: Invariant Relation

$$g^2 + 3n^2 = r^2 \tag{5}$$

which can be interpreted geometrically.

- The vertical line NN' will intersect the incircle sphere at two points N_3 and N_3' such that $NN_3 = NN_3' = n\sqrt{3}$. This is because $IN_3^2 = r^2 = IN^2 + NN_3^2 = g^2 + 3n^2$.
- We draw, in the plane of the base triangle, the tangents to the incircle from the radical origin point, calling F_t and F_t' the points of contact (fig. 12, p. 24).

Now because N and F_l are inverse points we have $F_tF_t^{'} \perp IN$ at N. If N'' is the orthocentre of the orthocentric tetrahedron drawn on the other side of the base plane, the triangles $F_tN'N''$ and $F_t^{'}N'N''$ are equilateral while the quadrilateral $N'F_tN''F_t^{'}$ is a $\frac{\pi}{3}$ rhombus. This is a kind of Morley's Theorem for the orthocentric tetrahedron of a triangle[2].

Remark 6.1 *Note that*

$$IN^2 = g^2 = (x_I - x_N)^2 + (y_I - y_N)^2 = (x - x_N)^2 + (r - y_N)^2 .$$

The coordinates of the Gergonne point are: $N(x_N = \frac{cz+by\cos A}{xz+yx+zy}x, y_N = \frac{xy}{xz+yx+zy}b\sin A)$ *but even replacing* $\sin A, \cos A$ *by*

$$\sin A = \frac{2rx}{x^2+r^2} \tag{6}$$

$$\cos A = \frac{x^2-r^2}{x^2+r^2} \tag{7}$$

does not give a symmetric equation in x, y, z *because one side of the triangle is taken along the horizontal axis. Trilinear coordinates would seem better suited but do not shorten the computation for the length of the segment. In the plane we would need to use a CAS software or a very lengthy computation to find*

$$IN^2 = g^2 = \frac{t}{s\,p^2}\left(p^2 - 3st\right) = \frac{t}{s} - 3\frac{t^2}{p^2} = r^2 - 3\frac{t^2}{p^2}.$$

With the same data we can compute the radius, R' of the circumsphere from the formula $\rho^2 = R^{'2} - m^2$ given by Court in [2, §823 p. 271], where ρ is the imaginary radius of the polar sphere and m the length of a bimedian. With $\rho = i\sqrt{2}n$ and m given by (2), we get

Corollary 6.1 *The radius of the circumsphere is given by*

$$R^{'2} = m^2 - 2n^2.$$

6.1 Volume of the tetrahedron

The volume is given by $V = \frac{1}{3}h\Delta^{'}$ where $\Delta^{'}$ is the area of the upper face. Now the upper face projects as the base triangle therefore its area is such that $\Delta = \Delta^{'} \cdot \sin\theta$ and because $h = f\cos\theta$ and $f \cdot g = r^2$ it yields

$$V = \frac{1}{3}h\Delta^{'} = \frac{1}{3}h\,\frac{\Delta}{\sin\theta} = \frac{1}{3}f\cos\theta\,\frac{\Delta}{\sin\theta} = \frac{1}{3}\,\frac{r^2}{g}\,\frac{\Delta}{\tan\theta} = \frac{1}{3}\,\frac{r^2\Delta}{n}.$$

[2] As a coincidence, from the Mathematics Genealogy Project, the author is a descendant of Frank Morley!

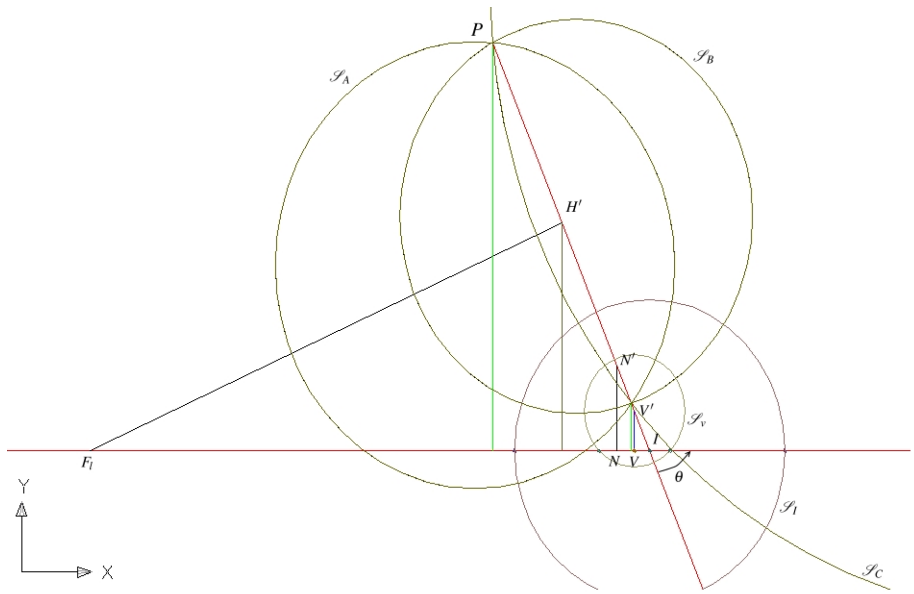


Figure 10: Cross-section along the principal plane

This leads to the simple formula

$$V = \frac{1}{3} r^3 \frac{s\,p}{t} = \frac{1}{3} p \sqrt{\frac{t}{s}} = \frac{1}{3} r\,p.$$

Remark 6.2 *In the special case where the triangle is equilateral we have* $s = 3x, p = 3x^2, t = x^3$ *so that*

$$V = \frac{1}{3} r^3 \frac{3x\,3x^2}{x^3} = 3r^3.$$

Another special case is when $p^2 - 4\,t\,s = 0$ *(see Remark 7.1) which yields*

$$V = \frac{1}{3} p \sqrt{\frac{t}{s}} = \frac{2}{3} t.$$

7 The common point of the six radical planes of the four orthogonal spheres.

The three vertex spheres centered on A', B', C' and the tri-tangent sphere (mutually orthogonal) have a radical centre, R, the common point of their six radical planes [2, p. 202 § 627], which has to be on the 4th altitude. The orthogonal projection of this point on the horizontal plane will therefore be on the Soddy line. We prove that this point is one of Eppstein's new triangle centre, denoted by M, the one giving the inner Soddy circle [7] (fig. 11).

We write $IM = m$ and m is given by Eppstein [7, Theorem 1 p. 65] as a barycentric quantity

$$m = \frac{\frac{p}{t}g + \frac{1}{\sigma}v}{\frac{p}{t} + \frac{1}{\sigma}} = \frac{\frac{g}{n} + \frac{v}{\sigma}}{\frac{1}{n} + \frac{1}{\sigma}}. \tag{8}$$

The radii of the Soddy circles, σ for the inner one, σ' for the outer one comes from (3). They satisfy , using $\sum x^2 \cdot y^2 = p^2 - 2ts$,

$$2\left(\frac{1}{\sigma^2} + \frac{\sum x^2 \cdot y^2}{x^2 y^2 z^2}\right) = 2\left(\frac{1}{\sigma^2} + \frac{p^2 - 2ts}{t^2}\right) = \left(\frac{1}{\sigma} + \frac{p}{t}\right)^2.$$

Expanding and simplifying yields

$$\frac{2}{\sigma^2} + 2\frac{p^2}{t^2} - 4\frac{s}{t} = \frac{1}{\sigma^2} + \frac{2}{\sigma}\frac{p}{t} + \frac{p^2}{t^2},$$

so that the reciprocal of σ and σ' are the roots of of

$$\frac{1}{\sigma^2} - \frac{2}{\sigma}\frac{p}{t} + \frac{p^2}{t^2} - 4\frac{s}{t} = 0 \Rightarrow$$

$$\frac{1}{\sigma} = \frac{p}{t} + 2\sqrt{\frac{s}{t}} = \frac{1}{n} + \frac{2}{r} \text{ and } \frac{1}{\sigma'} = \frac{p}{t} - 2\sqrt{\frac{s}{t}} = \frac{1}{n} - \frac{2}{r}. \tag{9}$$

We always have $0 < \sigma < |\sigma'|$. The radius σ' being positive if the contact of the outer Soddy circle is external, negative if internal.

Remark 7.1 *The limit case, for the sign of σ' is given by the equality $p^2 - 4ts = 0$. This is equivalent to the case $4R + r = 2s$ of [1] because if R is the radius of the circumcircle, we have*

$$R = \frac{\Delta}{4}\left(\frac{p}{t} - \frac{1}{s}\right) = \frac{\Delta}{4}\left(\frac{1}{n} - \frac{1}{s}\right).$$

In this case, it is well known that $\frac{1}{\sigma'} = 0$ and $g = \frac{r}{2}$. It yields $\cot\theta = 1$ therefore the angle between the Soddy line of the triangle and the fourth altitude of its tetrahedron is $\frac{\pi}{4}$.

With our notation, we also have

$$0 < \theta \leq \frac{\pi}{2} \Rightarrow \cot\theta = \frac{g}{n} = \frac{v}{\sigma} \Rightarrow v = \sigma\frac{g}{n}, \tag{10}$$

therefore (8) becomes

$$m = \frac{2\cot\theta}{\frac{2}{r} + \frac{2}{n}} = \frac{nr}{r+n}\cot\theta = \frac{rg}{r+n}.$$

Now, on the fourth altitude the point R is such that

$$RP \cdot RQ = RH'^2 - H'P^2 = RV'^2 - 2\sigma^2.$$

We note $IR = \rho$ and, with the origin at I, we get

$$(\rho - h)^2 - H'P^2 = (\rho - IV')^2 - 2\sigma^2$$

so that

$$\rho^2 - 2h\rho + h^2 - H'P^2 = \rho^2 - 2\rho\frac{v}{\cos\theta} + \frac{v^2}{\cos^2\theta} - 2\sigma^2.$$

In this equation, we have

$$h^2 - H'P^2 = r^2 \tag{11}$$

because it is the power of I relative to the vertex spheres, it yields

$$r^2 - 2h\rho = \frac{v^2}{\cos^2\theta} - 2\sigma^2 - 2\rho\frac{v}{\cos\theta}$$

and, in order to solve for $\rho\cos\theta$,

$$2\rho\left(h - \frac{v}{\cos\theta}\right) = r^2 - \frac{v^2}{\cos^2\theta} + 2\sigma^2,$$

we substitute $h\cos\theta = f\cos^2\theta = \frac{r^2}{g}\cos^2\theta$ therefore

$$\rho\cos\theta = \frac{g}{2}\frac{(r^2 + 2\sigma^2)\cos^2\theta - v^2}{r^2\cos^2\theta - v\,g}.$$

We have, combining (10) and (9)

$$\cot\theta = \frac{g}{n} = \frac{v}{\sigma} \Rightarrow \frac{g}{v} = \frac{1}{\sigma}n = 2\frac{n}{r} + 1 \Leftrightarrow \frac{r}{\sigma} = 2 + \frac{r}{n}.$$

Therefore, using the invariant relation (5),

$$\frac{1}{\cos^2\theta} = 1 + \tan^2\theta = 1 + \frac{n^2}{g^2} = \frac{g^2 + n^2}{g^2} = \frac{r^2 - 2n^2}{g^2}.$$

We express all these quantity as functions of the ratios $\frac{1}{\sigma}, \frac{r}{\sigma}, \frac{1}{\cos\theta}, \frac{g}{v}$ dividing numerator and denominator of $\rho\cos\theta$ by $\sigma^2 \cdot \cos^2\theta$. We get

$$\rho\cos\theta = \frac{g}{2} \cdot \frac{(r^2 + 2\sigma^2)\cos^2\theta - v^2}{r^2\cos^2\theta - v\,g} = \frac{g}{2}\frac{\left(\frac{r^2}{\sigma^2} + 2\right) - \frac{1}{\sigma^2}\frac{v^2}{\cos^2\theta}}{\frac{r^2}{\sigma^2} - \frac{1}{\sigma^2}\frac{v\,g}{\cos^2\theta}}$$

$$\rho\cos\theta = \frac{g}{2} \cdot \frac{\left(2 + \frac{r}{n}\right)^2 + 2 - \left(\frac{2}{r} + \frac{1}{n}\right)^2 \frac{v^2}{g^2}\left(r^2 - 2n^2\right)}{\left(2 + \frac{r}{n}\right)^2 - \left(\frac{2}{r} + \frac{1}{n}\right)^2 \frac{v}{g}\left(r^2 - 2n^2\right)},$$

substituting $\frac{v}{g} = \left(1+2\frac{n}{r}\right)^{-1}$ yields

$$\rho\cos\theta = \frac{g}{2}\cdot\frac{\left(2+\frac{r}{n}\right)^2+2-\left(\frac{2}{r}+\frac{1}{n}\right)^2\left(1+2\frac{n}{r}\right)^{-2}\left(r^2-2n^2\right)}{\left(2+\frac{r}{n}\right)^2-\left(\frac{2}{r}+\frac{1}{n}\right)^2\left(1+2\frac{n}{r}\right)^{-1}\left(r^2-2n^2\right)}.$$

Substituting the quantities $2+\frac{r}{n} = \left(1+2\frac{n}{r}\right)\frac{r}{n}$ and $\left(\frac{2}{r}+\frac{1}{n}\right) = \frac{1}{n}\left(1+2\frac{n}{r}\right)$ with some final simplification gives

$$\rho\cos\theta = \frac{r\cdot g}{r+n} = m.$$

We conclude that the orthogonal projection of R on the horizontal plane is M.

7.1 The outer Soddy circle

We extend to the outer Soddy circle, centered at W, and to a sphere, $\mathscr{S}_W$, carrying it the former properties (fig. 11). As for the tritangent sphere (cf. § 2, p. 4), we call the *outer tritangent sphere*, centre W', the sphere corresponding to the outer Soddy circle and tangent to the three auxiliary cones. Note that this one will have a real curve of intersection with the three cones.

To construct the outer tritangent sphere,

- Build the upper half of the three cones (vertices A', B', C').
- Use the outer tritangent circle and its contact point given by the Eppstein's construction [8, § III p. 20], it yields the three generatrices of the upper half cones whose intersection W'' gives the equivalent of V'' for the upper half cone.
- Therefore the vertical line through the centre of the outer tritangent circle goes through W'', and its intersection with the fourth altitude gives the centre W' of the outer tritangent sphere.

Remark 7.2 *In the degenerate case of remark 7.1, we know that the outer Soddy circle becomes the Gergonne line. Here it means that the outer tritangent sphere becomes the upper face of the tetrahedron. Hence this face is tangent to the three auxiliary cones and its angle with the horizontal plane is* $\frac{\pi}{4}$.

This sphere is also mutually orthogonal with the three vertex spheres and we call R' their radical centre.

It allows us to extend the proof of theorem 5.1 to the outer Soddy circle.

Theorem 7.1 *The Gergonne line is the radical axis of the coaxal pencil of circles formed by the incircle and both Soddy circles.*

The radius of this sphere is $\sqrt{2}\sigma'$ if σ' is the radius of the outer Soddy circle. Its value is given by (9)

$$\frac{1}{\sigma'} = \frac{1}{n} - \frac{2}{r}$$

so that, with the given orientation, the relation (10) becomes

$$\cot\theta = \frac{g}{n} = \frac{w}{\sigma'} \Rightarrow w = \sigma' \frac{g}{n}$$

and, for the corresponding Eppstein point $M^{'}$, the abscissa m' of $M^{'}$ is given by

$$m' = \frac{\frac{p}{t}g + \frac{1}{\sigma'}w}{\frac{p}{t} + \frac{1}{\sigma'}} = \frac{\frac{g}{n} + \frac{w}{\sigma'}}{\frac{1}{n} + \frac{1}{\sigma'}} = \frac{2\cot\theta}{\frac{2}{n} - \frac{2}{r}} = \frac{n\,r}{r-n}\cot\theta = \frac{r\,g}{r-n}.$$

$$m' = \frac{r \cdot g}{r-n}.$$

Hence, substituting $-n$ for n, we can use the same computation for the outer Eppstein point.

We have proved

Theorem 7.2 *The orthogonal projection on the Soddy line of the radical centres of the three vertex spheres and one of the tritangent sphere, inner or outer, is the corresponding Eppstein point of the base triangle.*

In [7, Theorem 1 p. 65], the harmonicity of the Eppstein points, M, M' with the pair I and N is mentioned as a Mathematica calculation. With our notations, we immediately get

$$(I,N,M,M') = \frac{IM}{IM'} : \frac{NM}{NM'} = \frac{m}{m'} : \frac{m-g}{m'-g} = -1.$$

From there, by projection, we find the harmonicity of the two radical centres, R, R' (three vertex spheres and one of the tritangent sphere, inner or outer), with the orthocentre of the orthocentric tetrahedron of a triangle and its incentre: $(I,N',R,R') = -1$.

Similarly we have

$$(I,N,V,W) = \frac{IV}{IW} : \frac{NV}{NW} = \frac{v}{w} : \frac{v-g}{w-g} = -1 \Rightarrow (I,N',V',W') = -1,$$

the incentre of the base triangle, the orthocentre of the tetrahedron and the centres of both tritangent spheres are harmonic.

8 Conclusion

This discovery is a by-product of the author's Ph.D. dissertation [11] about Voronoï's diagrams of circles. In it, the computation of the vertices of the diagram are done considering a Voronoï's diagrams of circles as the top view of a composite surface formed by cones of revolutions centered on the circles (cf. §2). With this point of view, the centre V of the inner Soddy circle is nothing but the vertex of the Voronoï's diagrams of the three circles $\mathscr{C}_A$, $\mathscr{C}_B$ and $\mathscr{C}_C$.

In a subsequent article, we will show that there is a one to one correspondence between a face of an orthocentric tetrahedron and a base triangle, provided the orthocentre is inside the tetrahedron, and we will study what happens if we iterate the process.

Moreover Richard Guy has already extended the construction of the tetrahedron to the excircles [9], adding some new properties in the process, so that this tetrahedron seems to have numerous geometrical properties worth exploring.

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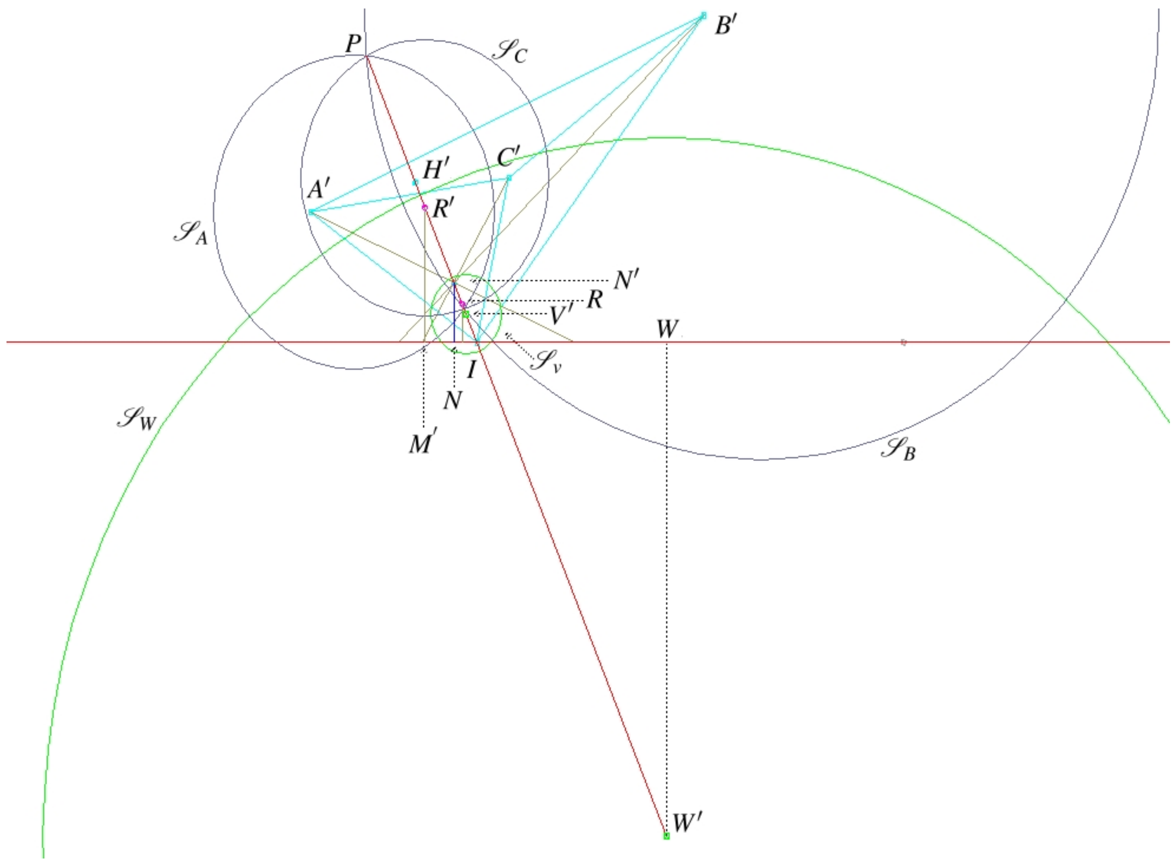

Figure 11: Traces on the principal plane of the 5 spheres

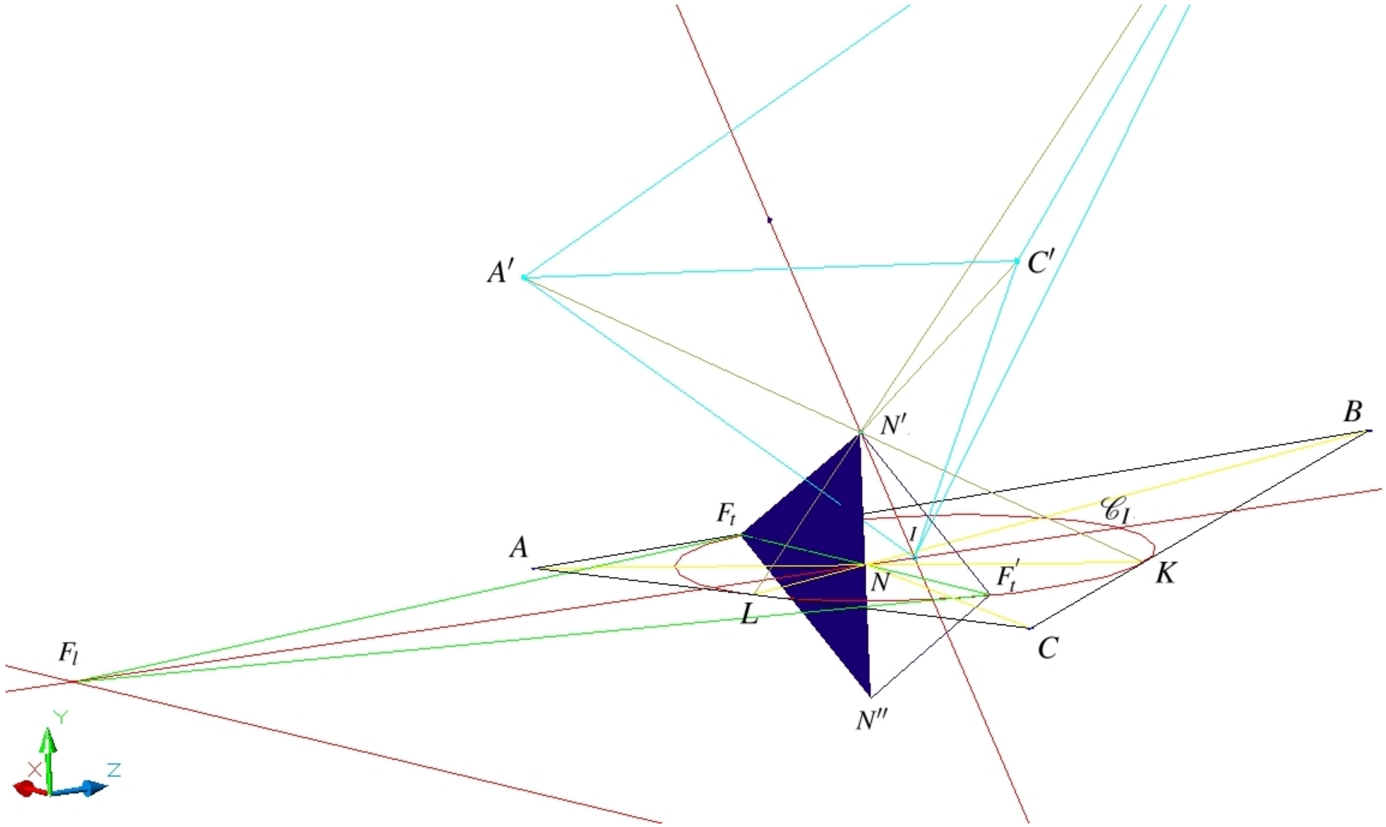

Figure 12: The equilateral triangle $N'N''F_t$ and the $\frac{\pi}{3}$ Rhombus $N'F_tN''F_t'$